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Document Version
Publisher's PDF, also known as Version of record

Publication date:
2004

Link to publication in University of Groningen/UMCG research database

Citation for published version (APA):
Klein Haneveld, W. K., Stougie, L., & van der Vlerk, M. H. (2004). *Simple Integer Recourse Models: Convexity and Convex Approximations*. s.n.

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Simple Integer Recourse Models: Convexity and Convex Approximations

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October 1, 2004

Abstract

We consider the objective function of a simple recourse problem with fixed technology matrix and integer second-stage variables. Separability due to the simple recourse structure allows to study a one-dimensional version instead.

Based on an explicit formula for the objective function, we derive a complete description of the class of probability density functions such that the objective function is convex. This result is also stated in terms of random variables.

Next, we present a class of convex approximations of the objective function, which are obtained by perturbing the distributions of the right-hand side parameters. We derive a uniform bound on the absolute error of the approximation. Finally, we give a representation of convex simple integer recourse problems as continuous simple recourse problems, so that they can be solved by existing special purpose algorithms.

Key words: simple integer recourse, convexity, convex approximation

Mathematics Subject Classification: 90C15, 90C11

SOM theme A: *The human and technical side of production: the management of interdependencies*

Electronic version: `http://som.rug.nl`

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* Research of this author has been made possible by a fellowship of the Royal Netherlands Academy of Arts and Sciences.

1. Introduction

The *simple integer recourse* (SIR) model with fixed technology matrix is defined as

$$\inf_x \left\{ cx + \mathcal{Q}(x) : Ax = b,\ x \in \mathbb{R}^{n_1}_+ \right\}, \tag{1}$$

where the *expected value function* $\mathcal{Q}$ is

$$\mathcal{Q}(x) := \mathbb{E}_\xi \left[v(\xi - Tx) \right],$$

with $\mathbb{E}_\xi[\cdot]$ denoting mathematical expectation with respect to ξ, and v is the value function of the second stage problem

$$\begin{aligned} v(s) := \inf_y \{ q^+ y^+ + q^- y^- : \quad & y^+ \geq s, \\ & y^- \geq -s, \\ & y^+, y^- \in \mathbb{Z}^{m_2}_+ \}, \qquad s \in \mathbb{R}^{m_2}. \end{aligned}$$

Here $c, A, b, (q^+, q^-)$ and T are vectors/matrices of the appropriate size, $q^+, q^- \geq 0$, $q^+ + q^- > 0$, and ξ is a random vector in $\mathbb{R}^{m_2}$.

As suggested by the name, this model has the same structure as the well-known simple *continuous* recourse model, in which the second-stage decision variables $y = (y^+, y^-)$ are non-negative reals. The expected value function of the latter problem is Lipschitz continuous and convex, so that in principle (i.e., disregarding possible difficulties in evaluating the integrals) it can be solved efficiently by standard techniques from mathematical programming, see [9]. Several special purpose algorithms exist, see e.g. [18, 1, 2, 10, 11]. The expected value function of the integer problem lacks these favorable properties in general. In [7] and [12] (the latter treating complete mixed-integer recourse) it is shown that this function is continuous if the distribution of ξ is continuous. In [3, 4] we discussed the construction of the convex hull of the discontinuous expected value function $\mathcal{Q}$ for the simple integer recourse model with discretely distributed right-hand side vector ξ. For an overview of the field of stochastic integer programming we refer to [5, 6, 14] and the website [13].

In the first part of this paper we give a complete description of the class of distributions such that the expected value function $\mathcal{Q}$ is convex. Following a review of relevant results in Section 2, this class will be presented in Section 3.

Using separability which is due to the simple recourse structure, $\mathcal{Q}$ is completely characterized by the one-dimensional generic function Q, given by

$$Q(z) := q^+ g(z) + q^- h(z), \quad z \in \mathbb{R},$$

where $q^+, q^- \in \mathbb{R}$, with $q^+ \geq 0$, $q^- \geq 0$, $q^+ + q^- > 0$,

$$\begin{aligned} g(z) &:= \mathbb{E}_\xi \left[\lceil \xi - z \rceil^+ \right], & z \in \mathbb{R}, \\ h(z) &:= \mathbb{E}_\xi \left[\lfloor \xi - z \rfloor^- \right], & z \in \mathbb{R}, \end{aligned}$$

with ξ a random variable, and $\lceil s \rceil^+$ and $\lfloor s \rfloor^-$ are shorthand notations for $(\lceil s \rceil)^+ := \max\{0, \lceil s \rceil\}$ and $(\lfloor s \rfloor)^- := \max\{0, -\lfloor s \rfloor\}$, $s \in \mathbb{R}$, respectively. Hence, we will derive convexity results for

$\mathcal{Q}$ by doing so for Q.

Next, using the results obtained so far, we present a class of convex approximations of Q. In Section 6 we concentrate on the integer *expected surplus function* g first, and then extend the results to the *expected shortage function* h and the one-dimensional expected value function Q. Moreover, using a result from [3], we show in Section 7 that (up to a constant) the approximation of Q is the expected value function of a continuous simple recourse problem, with random right-hand side parameter with explicitly given discrete distribution. In Section 8 these results are extended to the n_1-dimensional expected value function $\mathcal{Q}$. Finally, in Section 9, we summarize our results and indicate their relevance for problems with more general recourse structures.

2. Preliminaries

In this section we first review some structural properties of the expected value function of the simple integer recourse problem. Subsequently we present some preliminary lemmas.

2.1 Review

Some of the results in this subsection are quoted from [7] where the reader is referred to for proofs.

Let F and $\hat{F}$ be respectively the right- and left-continuous cumulative distribution function (cdf) of the random variable ξ, i.e., $F(s) := \Pr\{\xi \le s\}$ and $\hat{F}(s) := \Pr\{\xi < s\}$. (Obviously, if ξ has a probability density function (pdf) then $F \equiv \hat{F}$.) Then (cf. [7])

$$g(z) = \sum_{k=0}^{\infty}(1 - F(z+k)) = \sum_{k=0}^{\infty}\Pr\{\xi > z+k\} \tag{2}$$

$$h(z) = \sum_{k=0}^{\infty}\hat{F}(z-k) = \sum_{k=0}^{\infty}\Pr\{\xi < z-k\}. \tag{3}$$

The functions g and h are related by an elementary transformation.

Lemma 2.1 *Let ξ be a random variable. Define $\zeta = -\xi$. Then*

$$h(z) = g^{\zeta}(-z), \quad z \in \mathbb{R},$$

where $g^{\zeta}(z) := \mathbb{E}_{\zeta}\left[\lceil \zeta - z\rceil^{+}\right]$.

The random variable ζ has cdf $F_{\zeta}(s) = 1 - \hat{F}(-s)$. If ξ has a pdf f then ζ has a pdf $f_{\zeta}(s) = f(-s)$.

PROOF. Since $\lfloor s\rfloor^{-} = \lceil -s\rceil^{+}$, $s \in \mathbb{R}$, it follows that

$$h(z) = \mathbb{E}_{\xi}\left[\lfloor \xi - z\rfloor^{-}\right] = \mathbb{E}_{\xi}\left[\lceil -\xi + z\rceil^{+}\right] = \mathbb{E}_{\zeta}\left[\lceil \zeta + z\rceil^{+}\right] = g^{\zeta}(-z), \quad z \in \mathbb{R}.$$

The relations between the cdf (and pdf) of the random variables ξ and ζ are trivial. $\square$

Finiteness of the various functions is directly related to finiteness of the first moment of ξ.

Lemma 2.2 *For all $z \in \mathbb{R}$*

$$\begin{aligned} g(z) < \infty \quad &\Leftrightarrow \quad \mu^+ = \mathbb{E}_\xi\left[(\xi)^+\right] < \infty; \\ h(z) < \infty \quad &\Leftrightarrow \quad \mu^- = \mathbb{E}_\xi\left[(\xi)^-\right] < \infty; \\ Q(z) < \infty \quad &\Leftrightarrow \quad -\infty < \mu = \mu^+ - \mu^- < \infty. \end{aligned}$$

From now on we assume that μ is finite.

Lemma 2.3 *The functions g, h, and Q are continuous on $\mathbb{R}$ if and only if the random variable ξ is continuously distributed.*

2.2 Right derivatives

In studying convexity of the functions g, h, and Q we will use a necessary and sufficient condition for convexity of a function in terms of its right derivative: a function φ is convex on the interval $[a, b]$ if and only if its right derivative φ'_+ is non-decreasing on $[a, b)$.

As we will see, existence of the right derivatives of g, h, and Q depends on the total variation of the density f of ξ. Although we only need finiteness of the series involved we will prove sharp lower and upper bounds. The reason is that these bounds are essential for obtaining an error bound for the convex approximations that we have in mind.

For φ a real-valued function on a non-empty subset I of $\mathbb{R}$, we denote the *total increase* on I by $\Delta^+\varphi(I)$, the *total decrease* by $\Delta^-\varphi(I)$, and the *total variation* by $|\Delta|\varphi(I)$. For all $I \subset \mathbb{R}$, $|\Delta|\varphi(I) = \Delta^+\varphi(I) + \Delta^-\varphi(I)$. For convenience we use the shorthand notations $\Delta^+\varphi$, $\Delta^-\varphi$, and $|\Delta|\varphi$ for the case $I = \mathbb{R}$. A real function φ on $\mathbb{R}$ is *of bounded variation* if $|\Delta|\varphi < +\infty$.

We establish a result for probability density functions of bounded variation. It is based on the following lemma, which is proved in Appendix 9.1

Lemma 2.4 *Let the real functions φ_i on $\mathbb{R}_+$ be nonnegative, nonincreasing and integrable, $i = 1, 2$. Then the function $\varphi := \varphi_1 - \varphi_2$ is of bounded variation on $[0, \infty)$. Moreover,*

$$-\infty < -\Delta^+\varphi([0,\infty)) \leq \sum_{k=0}^{\infty} \varphi(k) - \int_0^\infty \varphi(s)\,ds \leq \Delta^-\varphi([0,\infty)) < \infty \tag{4}$$

and

$$-\infty < -\Delta^-\varphi([0,\infty)) \leq \sum_{k=1}^{\infty} \varphi(k) - \int_0^\infty \varphi(s)\,ds \leq \Delta^+\varphi([0,\infty)) < \infty. \tag{5}$$

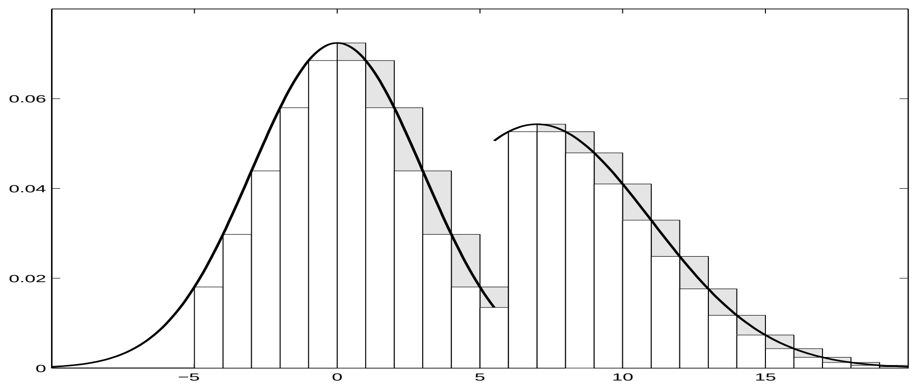


Figure 2.1: Illustration of the upper bound (6) in Lemma 2.5 with $z = -5$. Each term $f(z+k)$, $k = 0, 1, \ldots$, is represented by a rectangle of size $f(z+k) \times 1$. The total area of the rectangles is divided in two: the unshaded area is less than $1 - F(z)$, and the shaded area is less than $\Delta^- f([z, \infty))$.

In the following lemma we apply these results to a class of probability density functions on $\mathbb{R}$ that are of bounded variation.

Lemma 2.5 *Suppose that for $i = 1, 2$ the functions $f_i : \mathbb{R} \mapsto [0, 1]$ satisfy the following conditions:*

- *(i) f_i is nondecreasing, $f_i(-\infty) = 0$, $f_i(+\infty) = 1$, $i = 1, 2$.*
- *(ii) $f_1(s) \geq f_2(s)$ for all $s \in \mathbb{R}$, and $\{s \in \mathbb{R} : f_1(s > f_2(s))\}$ has a positive Lebesgue measure.*
- *(iii) $\int_{-\infty}^0 f_1(s)\,ds < \infty$ and $\int_0^\infty (1 - f_2(s))\,ds < \infty$.*

Then $c := \int_{-\infty}^\infty (f_1(s) - f_2(s))\,ds \in (0, \infty)$, and the function $f : \mathbb{R} \mapsto \mathbb{R}_+$ defined by

$$f(s) = \frac{1}{c}\left(f_1(s) - f_2(s)\right), \quad s \in \mathbb{R},$$

is a pdf. Moreover, f has a right-continuous version f_+ and a left-continuous version f_-, given by

$$\begin{aligned} f_+(s) &:= \lim_{t \downarrow s} f(t), \quad s \in \mathbb{R}, \\ f_-(s) &:= \lim_{t \uparrow s} f(t), \quad s \in \mathbb{R}, \end{aligned}$$

having the same cumulative distribution function as f. Denoting this cdf by F, we have for all $z \in \mathbb{R}$

$$1 - F(z-1) - \Delta^- f([z-1, \infty)) \leq \sum_{k=0}^{\infty} f(z+k) \leq 1 - F(z) + \Delta^- f([z, \infty)) \qquad (6)$$

and

$$F(z) - \Delta^+ f((-\infty, z]) \le \sum_{k=1}^{\infty} f(z-k) \le F(z-1) + \Delta^+ f((-\infty, z-1]). \tag{7}$$

The pdf f is of bounded variation, so that the following uniform bounds hold, too:

$$1 - F(z-1) - \frac{|\Delta| f}{2} \le \sum_{k=0}^{\infty} f(z+k) \le 1 - F(z) + \frac{|\Delta| f}{2} \tag{8}$$

$$F(z) - \frac{|\Delta| f}{2} \le \sum_{k=1}^{\infty} f(z-k) \le F(z-1) + \frac{|\Delta| f}{2}. \tag{9}$$

Remark 2.1 The conditions on f_1 and f_2 are not very strong, if one wants to describe pdfs that are of bounded variation. Indeed, any non-constant function f of bounded variation that converges to 0 at the tails, can be written as the difference $\tilde{f}_1 - \tilde{f}_2$ of nondecreasing functions $\tilde{f}_1$ and $\tilde{f}_2$ with $a := \tilde{f}_1(-\infty) = \tilde{f}_2(-\infty)$ and $b := \tilde{f}_1(\infty) = \tilde{f}_2(\infty)$, $-\infty < a < b < \infty$. Then

$$f(s) = \tilde{f}_1(s) - \tilde{f}_2(s) = b\left(f_1(s) - f_2(s)\right), \quad s \in \mathbb{R},$$

where $f_i(s) = (\tilde{f}_i(s) - a)/(b-a)$, $i = 1, 2$, satisfy (i). In order to assure that f is a pdf, obviously the condition (ii) is needed together with $b = 1/c$. Hence, the only real assumption is (iii), describing that f_1 and f_2 converge 'fast enough' at their tails.

PROOF. Obviously, $c > 0$ because of (ii). Moreover, $c < \infty$ because of (i) and (iii):

$$\begin{aligned}\int_{-\infty}^{\infty} \left(f_1(s) - f_2(s)\right) ds &= \int_{-\infty}^{0} \left(f_1(s) - f_2(s)\right) ds + \int_{0}^{\infty} \left(f_1(s) - f_2(s)\right) ds \\ &\le \int_{-\infty}^{0} f_1(s)\, ds + \int_{0}^{\infty} \left(1 - f_2(s)\right) ds < \infty.\end{aligned}$$

Therefore, f is a pdf. Moreover, since f_1 and f_2 are monotonous, they have all limits from left and right, hence so does f. Since f_1 and f_2 are nonincreasing and bounded, they can only have a countable number of discontinuities, so that $f_-(s) = f(s) = f_+(s)$ for all $s \in \mathbb{R}$ except for a set of measure 0. The upper bound in (6) follows from the upper bound in (4) by taking, for $s \in \mathbb{R}_+$,

$$\varphi(s) = f(z+s),\ \varphi_1(s) = \frac{1 - f_2(z+s)}{c} \text{ and } \varphi_2(s) = \frac{1 - f_1(z+s)}{c}.$$

Both φ_1 and φ_2 are nonnegative, nonincreasing and integrable on $\mathbb{R}_+$ since $0 \le \int_z^{\infty}(1 - f_1(s))\, ds \le \int_z^{\infty}(1 - f_2(s))\, ds < \infty$. The lower bound in (6) follows from the lower bound in

(5) by taking

$$\varphi(s) = f(z-1+s),\ \varphi_1(s) = \frac{1-f_2(z-1+s)}{c} \text{ and } \varphi_2(s) = \frac{1-f_1(z-1+s)}{c},$$

using $\int_z^\infty (1-f_1(s))\,ds \le \int_z^\infty (1-f_2(s))\,ds < \infty$. Similarly, the upper and lower bound in (7) follow from (4) and (5), by taking, for $s \in \mathbb{R}$,

$$\varphi(s) = f(z-1-s),\ \varphi_1(s) = \frac{1-f_2(z-1-s)}{c} \text{ and } \varphi_2(s) = \frac{1-f_1(z-1-s)}{c},$$

and

$$\varphi(s) = f(z-s),\ \varphi_1(s) = \frac{1-f_2(z-s)}{c} \text{ and } \varphi_2(s) = \frac{1-f_1(z-s)}{c},$$

respectively. Finally, (8) and (9) follow from (6) and (7) by observing that, for all $z \in \mathbb{R}$, $\Delta^- f([z,\infty)) \le \Delta^- f$ and $\Delta^+ f((-\infty,z]) \le \Delta^+ f$. Indeed, since $\Delta^+ f - \Delta^- f = f(\infty) - f(-\infty) = 0 - 0 = 0$ and $\Delta^+ f + \Delta^- f = |\Delta| f$, we have that $\Delta^- f = -\Delta^+ f = (|\Delta| f)/2$. □

See Figure 2.1 for an illustration of the upper bounds (6) c.q. (8).

With Lemma 2.5 in mind, let us define the following class of probability density functions.

Definition 2.1 Let $\mathcal{F}$ be the class of probability density functions on $\mathbb{R}$ with finite mean value, that can be written as the difference of functions f_1 and f_2 satisfying the conditions (i)–(iii) of Lemma 2.5.

It is easy to verify that $\mathcal{F}$ is a convex set.

As said before, from a practical point of view $\mathcal{F}$ contains all pdfs with a bounded variation. For instance, if f is a pdf (with $\int |s| f(s)\,ds < \infty$) such that $\mathbb{R}$ can be partitioned in a finite number of intervals on each of which f is either nondecreasing or nonincreasing, f must be in $\mathcal{F}$.

The final lemma of this subsection gives formulae for the right derivatives of the functions g, h, and $Q = q^+ g + q^- h$. They follow directly from (2) and (3) by interchanging differentiation and summation. This interchange is allowed since the sums converge uniformly in z (see [7] for more details).

Lemma 2.6 *Let ξ be a random variable with pdf $f \in \mathcal{F}$. Then the right derivative Q'_+ exists and is finite everywhere, and is given by*

$$Q'_+(z) = -q^+ \sum_{k=0}^{\infty} f_+(z+k) + q^- \sum_{k=0}^{\infty} f_+(z-k), \quad z \in \mathbb{R},$$

where f_+ is the right-continuous version of f given in Lemma 2.5.

Obviously, corresponding results for the constituting functions g and h are obtained by appropriate choices of q^+ and q^-.

3. Convexity properties

We will give a complete characterization of the pdfs in $\mathcal{F}$ such that the functions g, h, and Q are convex. There is no loss in assuming a continuous distribution, since Lemma 2.3 shows that if ξ is discretely distributed, then these functions are finite and discontinuous and hence non-convex. Also for continuous distributions of ξ convexity is an exception rather than the rule. However, for any distribution of ξ, the restriction of g, h, and Q to translates of $\mathbb{Z}$ is convex.

Lemma 3.1 *For every $\alpha \in [0, 1)$, the restriction of g, h, and Q to $\{\alpha + \mathbb{Z}\}$ is convex.*

PROOF. Using (2) we have

$$g(\alpha + n + 1) - g(\alpha + n) = F(\alpha + n) - 1 \quad \forall n \in \mathbb{Z}.$$

The result follows from the fact that the cdf F is non-decreasing. Using (3) similarly, the same result holds for h and Q. □

Next we formally define the set $\mathcal{C} \subset \mathcal{F}$ of probability density functions such that the corresponding expected value function Q (and hence both g and h) is convex. Due to Lemma 2.6 we have

Definition 3.1 Let $\mathcal{C}$ denote the set of probability density functions in $\mathcal{F}$ such that the corresponding expected value function Q is convex, i.e.,

$$\mathcal{C} = \left\{ f \in \mathcal{F} : \begin{array}{l} \sum_{k=0}^{\infty} f_+(z+k) \text{ is a nonincreasing function of } z, \text{ and} \\ \sum_{k=0}^{\infty} f_+(z-k) \text{ is a nondecreasing function of } z \end{array} \right\}.$$

The subset containing all right-continuous versions of elements of $\mathcal{C}$ is denoted by $\mathcal{C}_+$.

Given that $\mathcal{F}$ is convex, it is obvious that $\mathcal{C}$ and $\mathcal{C}_+$ are convex sets.

The following lemma gives a property shared by all elements of $\mathcal{C}$.

Lemma 3.2 *If $f \in \mathcal{C}$ then $\sum_{k=-\infty}^{\infty} f_+(z+k) = 1$ for all $z \in \mathbb{R}$.*

PROOF. By definition, $f \in \mathcal{C}$ if and only if Q is convex when the random variable ξ has pdf f.

It holds $\bar{Q}(z) \leq Q(z) \leq \bar{Q}(z) + \max\{q^+, q^-\}$ for all $z \in \mathbb{R}$, where the convex function $\bar{Q}$ is the one-dimensional expected value function of the continuous relaxation of (1), i.e., $\bar{Q}(z) := q^+\mathbb{E}_\xi\left[(\xi - z)^+\right] + q^-\mathbb{E}_\xi\left[(\xi - z)^-\right]$, $z \in \mathbb{R}$ (see [15]). Hence, if Q is convex, it has asymptotes at $-\infty$ and $+\infty$ with the same slopes as the asymptotes of $\bar{Q}$, which are $-q^+$ and q^-, respectively. Therefore, in this case Q'_+ is nondecreasing from $\lim_{z\to-\infty} Q'_+(z) = -q^+$

to $\lim_{z\to\infty} Q'_+(z) = q^-$. Using Lemma 2.6 we have, for $z \in \mathbb{R}$ and $n \in \mathbb{Z}$,

$$\begin{aligned}
Q'_+(z+n) &= -q^+ \sum_{k=n}^{\infty} f_+(z+k) + q^- \sum_{k=-\infty}^{n} f_+(z+k) \\
&= \sum_{k=-\infty}^{\infty} \left(-q^+ \cdot 1_{\{k\geq n\}} + q^- \cdot 1_{\{k\leq n\}}\right) f_+(z+k). \qquad (10)
\end{aligned}$$

Since $f \in \mathcal{F}$, it follows (see (8) and (9)) that

$$S(z) = \sum_{k=-\infty}^{\infty} f_+(z+k)$$

is finite for all $z \in \mathbb{R}$. Therefore we have, for all $n \in \mathbb{Z}$,

$$\sum_{k=-\infty}^{\infty} \left|-q^+ \cdot 1_{k\geq n} + q^- \cdot 1_{k\leq n}\right| f_+(z+k) \leq (q^+ + q^-)S(z) < \infty,$$

so that by taking $n \to -\infty$ and $n \to \infty$ in (10) we obtain, using Lebesgue's dominated convergence theorem, that

$$-q^+ = -q^+ S(z) \qquad \text{and} \qquad q^- = q^- S(z),$$

respectively, implying $S(z) = 1$. □

Note that the condition $\sum_{k=-\infty}^{\infty} f_+(z+k) = 1$ for all $z \in \mathbb{R}$ is necessary but not sufficient for $f \in \mathcal{C}$. A counter example is given by the pdf f such that

$$f_+(s) = \begin{cases} |s|, & s \in [-1, 1), \\ 0, & \text{otherwise.} \end{cases}$$

It is easily seen that $f \in \mathcal{F}$ and $\sum_{k=-\infty}^{\infty} f_+(z+k) = 1$ for all $z \in \mathbb{R}$. However, $f \notin \mathcal{C}$ since $\sum_{k=0}^{\infty} f_+(z+k)$ is strictly increasing on $[0, 1)$.

As said before, convexity of the functions g, h, and Q is an exception rather than the rule. Therefore, $\mathcal{C}$ is a 'small' subset of $\mathcal{F}$. In the following we first define a subset of $\mathcal{C}$, using a characterization that is attractive from a computational point of view. Next, we show that this subset actually contains all of $\mathcal{C}$.

Definition 3.2 Let $\mathcal{C}_0$ be the set of functions f on $\mathbb{R}$, that can be written as

$$f(s) = G(s+1) - G(s), \qquad s \in \mathbb{R},$$

where G is an arbitrary cdf on $\mathbb{R}$ with a finite mean value μ_G.

The following lemma shows that $\mathcal{C}_0$ is a subset of $\mathcal{C}$.

Lemma 3.3 *Let f be generated by a cdf G with mean value μ_G as described in Definition 3.2. Then f is continuous from the right, and the following is true.*

(i) For $s \in \mathbb{R}$

$$\sum_{k=0}^{\infty} f(s+k) = 1 - G(s), \quad \sum_{k=1}^{\infty} f(s-k) = G(s), \quad \text{and} \sum_{k=-\infty}^{\infty} f(s+k) = 1.$$

In particular, $f \in \mathcal{C}_+ \subset \mathcal{C}$.

(ii) If F denotes the cdf of f, and η a random variable distributed according to G, we have for all $z \in \mathbb{R}$

$$F(z) = \int_z^{z+1} G(s)\,ds$$

$$\sum_{k=0}^{\infty} \big(1 - F(z+k)\big) = \int_z^{\infty} \big(1 - G(s)\big)\,ds = \mathbb{E}_\eta\left[(\eta - z)^+\right]$$

$$\sum_{k=1}^{\infty} F(z-k) = \int_{-\infty}^{z} G(s)\,ds = \mathbb{E}_\eta\left[(\eta - z)^-\right].$$

(iii) If it exists, the nth moment of f, denoted by ν_n, is equal to

$$\nu_n = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} (-1)^{n-k} \tau_k,$$

where τ_k is the kth moment of G. In particular, the mean value μ_f of f equals $\mu_G - \frac{1}{2}$, and its variance σ_f^2 is equal to $\sigma_G^2 + \frac{1}{12}$, where σ_G^2 is the variance of G (possibly infinite).

PROOF. (i) Since $f_1(s) := G(s+1)$ and $f_2(s) := G(s)$ satisfy all conditions in Lemma 2.5, we conclude that $f \in \mathcal{F}$. Indeed,

$$\begin{aligned}\int_{-\infty}^{0} f_1(s)\,ds &= \int_{-\infty}^{1} G(s)\,ds = \int_{-\infty}^{1} \left\{ \int_{(-\infty,s]} dG(t) \right\} ds \\ &= \int_{(-\infty,1]} \left\{ \int_t^1 ds \right\} dG(t) = \int_{(-\infty,1]} (1-t)\,dG(t) < \infty\end{aligned}$$

since μ_G is finite. Similarly $\int_{-\infty}^{0}(1 - f_2(s))\,ds < \infty$. The corresponding value of c is equal to 1, since

$$\begin{aligned}\int_{-\infty}^{\infty} \big(G(s+1) - G(s)\big)\,ds &= \int_{-\infty}^{\infty} \left\{ \int_{(s,s+1]} dG(t) \right\} ds \\ &= \int_{-\infty}^{\infty} \left\{ \int_{t-1}^{t} ds \right\} dG(t) = 1.\end{aligned}$$

Moreover,

$$\sum_{k=0}^{N} f(s+k) = \sum_{k=0}^{N} \big(G(s+k+1) - G(s+k)\big) = G(s+N+1) - G(s),$$

so that $\sum_{k=0}^{\infty} f(s+k) = 1 - G(s)$. In a similar way we get $\sum_{k=1}^{\infty} f(s-k) = G(s)$, so that $\sum_{k=-\infty}^{\infty} f(s+k) = 1$ for all $s \in \mathbb{R}$. Since $f = f_+$ (because G is a cdf) it follows from the definition of $\mathcal{C}_+$ that $f \in \mathcal{C}_+$ indeed.

(ii) It is not difficult to see that $F(z) = \int_z^{z+1} G(s)\,ds$ for all $z \in \mathbb{R}$, so that

$$\sum_{k=1}^{\infty} F(z-k) = \sum_{k=1}^{\infty} \int_{z-k}^{z-k+1} G(s)\,ds = \int_{-\infty}^{z} G(s)\,ds = \mathbb{E}_\eta\left[(\eta - z)^-\right],$$

and similarly $\sum_{k=0}^{\infty}\left(1 - F(z+k)\right) = \mathbb{E}_\eta\left[(\eta - z)^+\right]$ for all $z \in \mathbb{R}$.

(iii) If it exists, the nth moment of f is given by

$$\begin{aligned}
\nu_n &= \int_{-\infty}^{\infty} s^n f(s)\,ds = \int_{-\infty}^{\infty} s^n \left\{\int_{(s,s+1]} dG(t)\right\} ds \\
&= \int_{-\infty}^{\infty} \left\{\int_{t-1}^{t} s^n\,ds\right\} dG(t) = \int_{-\infty}^{\infty} \frac{1}{n+1}\left(t^{n+1} - (t-1)^{n+1}\right) dG(t) \\
&= \int_{-\infty}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} (-1)^{n-k} t^k\,dG(t) \\
&= \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} (-1)^{n-k} \tau_k,
\end{aligned}$$

where τ_k is the kth moment of G. The expressions for μ_f and σ_f^2 follow by straightforward computation.

□

Theorem 3.1 $\mathcal{C}_+ = \mathcal{C}_0$.

PROOF. We already established that $\mathcal{C}_0 \subset \mathcal{C}_+$. To prove the reverse inclusion, suppose that $f \in \mathcal{C}_+$. Then we have to show that it can be represented as $f(s) = G(s+1) - G(s)$, $s \in \mathbb{R}$, where G is some cdf with finite mean value μ_G. We will show that $\bar{G}(s) := \sum_{k=1}^{\infty} f(s-k)$, $s \in \mathbb{R}$, meets the requirements.

Since f is continuous from the right,

$$\bar{G}(s) = \sum_{k=1}^{\infty} f(s-k) = \sum_{k=1}^{\infty} f_+(s-k) = \sum_{k=0}^{\infty} f_+(s-1-k), \quad s \in \mathbb{R},$$

so that $f \in \mathcal{C}$ implies that $\bar{G}$ is a nondecreasing function on $\mathbb{R}$.

Lemma 3.2 shows that

$$\begin{aligned}
\lim_{s\to\infty} \bar{G}(s) &= \lim_{n\to\infty} \bar{G}(n+1) = \lim_{n\to\infty} \sum_{k=-\infty}^{n} f(k) = \sum_{k=-\infty}^{\infty} f(k) = 1 \\
\lim_{s\to-\infty} \bar{G}(s) &= \lim_{n\to-\infty} \sum_{k=-\infty}^{n} f(k) = 0.
\end{aligned}$$

Moreover, for all $z \in \mathbb{R}$ we have

$$\lim_{s \downarrow z} \bar{G}(s) = \lim_{s \downarrow z} \sum_{k=1}^{\infty} f(s-k) = \sum_{k=1}^{\infty} \lim_{s \downarrow z} f(s-k) = \sum_{k=1}^{\infty} f(z-k) = \bar{G}(z),$$

where the inner equality is based on Lebesgue's dominated convergence theorem (using $\sum_{k=1}^{\infty} f(s-k) \leq 1 \ \forall s$). We conclude that $\bar{G}$ is a cdf indeed. Furthermore, it is easy to see that $f(s) = \bar{G}(s+1) - \bar{G}(s)$, $s \in \mathbb{R}$.

It remains to show that $\mu_{\bar{G}}$ is finite. This follows from

$$\begin{aligned} \mu_{\bar{G}} - 1/2 &= \int_{-\infty}^{\infty} (t - 1/2)\, d\bar{G}(t) = \int_{-\infty}^{\infty} \left\{ \int_{t-1}^{t} s\, ds \right\} d\bar{G}(t) \\ &= \int_{-\infty}^{\infty} s \left\{ \int_{(s,s+1]} d\bar{G}(t) \right\} ds = \int_{-\infty}^{\infty} s \left(\bar{G}(s+1) - \bar{G}(s) \right) ds \\ &= \int_{-\infty}^{\infty} s f(s)\, ds, \end{aligned}$$

which is finite since $f \in \mathcal{F}$. □

Corollary 3.1 *$f \in \mathcal{C}$ if and only if $G(s) = \sum_{k=1}^{\infty} f_+(s-k)$, $s \in \mathbb{R}$, is a cdf with finite mean value.*

PROOF. Without loss of generality we assume that $f = f_+$, since $f \in \mathcal{C}$ if and only $f_+ \in \mathcal{C}$. The result now follows trivially from Theorem 3.1 and its proof. □

We conclude from Corollary 3.1 that, disregarding $f \notin \mathcal{F}$, the function Q is convex if and only if the random variable ξ has a pdf f such that $\sum_{k=1}^{\infty} f_+(s-k)$ is a cdf with finite mean value.

Moreover, using Definition 3.2 and Theorem 3.1, we can approximate *any* given distribution (including discrete or mixed distributions) by a continuous distribution with a pdf $f \in \mathcal{C}$. Accordingly, we obtain a convex approximation of the expected value function Q, which we recall is non-convex in general. Because of Lemma 3.3 (iii) an interesting choice is to let f be generated by $G(s) = F(s - 1/2)$, where F is the cdf of the given distribution, since $f(s) = F(s+1/2) - F(s-1/2)$, $s \in \mathbb{R}$, has the same mean value as F (and a variance that is increased by $1/12$). We proceed by giving a few examples of cdfs F and corresponding pdfs $f \in \mathcal{C}$.

Example 3.1 Let F be a cdf and define $f(s) := F(s + 1/2) - F(s - 1/2)$, $s \in \mathbb{R}$.

(a) If F is the cdf of the degenerated distribution in 0 then

$$f(s) = \begin{cases} 1, & s \in [-1/2, 1/2), \\ 0, & \text{otherwise}, \end{cases}$$

i.e., f is the right-continuous pdf of the uniform distribution on $[-1/2, 1/2]$ (notation $\mathcal{U}(-1/2, 1/2)$).

(b) If F is the cdf of a discrete distribution on $\mathbb{Z}$ with $\Pr_F\{k\} = p_k \geq 0$, $\sum_{k \in \mathbb{Z}} p_k = 1$, such that $-\infty < \sum_{k\in\mathbb{Z}} p_k k < \infty$, then

$$f(s) = p_{\lfloor s+1/2 \rfloor},$$

i.e., f is the right-continuous pdf of a distribution which is uniform (with weight p_k) on every interval $[k - 1/2, k + 1/2)$, $k \in \mathbb{Z}$.

(c) If F is the cdf of the distribution $\mathcal{U}(0, 1)$ then

$$f(s) = \begin{cases} s + 1/2, & s \in [-1/2, 1/2), \\ 3/2 - s, & s \in [1/2, 3/2), \\ 0, & \text{otherwise,} \end{cases}$$

i.e., f is the right-continuous pdf of the triangular distribution on $[-1/2, 3/2]$.

(d) If F is the cdf of the distribution $\mathcal{U}(0, 1/2)$ then

$$f(s) = \begin{cases} 2s + 1, & s \in [-1/2, 0), \\ 1, & s \in [0, 1/2), \\ 2 - 2s, & s \in [1/2, 1), \\ 0, & \text{otherwise.} \end{cases}$$

(e) If F is the cdf of the exponential distribution with parameter λ then

$$f(s) = \begin{cases} 1 - e^{-\lambda(s+1/2)}, & s \in [-1/2, 1/2), \\ (e^{\lambda/2} - e^{-\lambda/2})e^{-\lambda s}, & s \in [1/2, \infty), \\ 0, & \text{otherwise.} \end{cases}$$

(f) If F is the cdf of the normal distribution with mean value μ and variance σ^2 then

$$f(s) = \frac{1}{\sqrt{2\pi}\sigma} \int_{s-1/2}^{s+1/2} e^{-\frac{(t-\mu)^2}{2\sigma^2}}\, dt, \quad s \in \mathbb{R}.$$

$\triangleleft$

The description of $\mathcal{C}$ is constructive in the sense that given a cdf G it is straightforward to construct the corresponding pdf $f \in \mathcal{C}$. On the other hand, for a given pdf f it is in general not easy to determine if it belongs to $\mathcal{C}$, i.e., to decide if $\sum_{k=1}^{\infty} f_+(s - k)$, $s \in \mathbb{R}$, is a cdf with finite mean value. The following two results settle this question for a restricted class of probability density functions.

Corollary 3.2 *Let the pdf f have support $[a, a + 1]$, $a \in \mathbb{R}$. Then $f \in \mathcal{C}$ if and only if f is a pdf of the uniform distribution on this interval.*

PROOF. It follows from Theorem 3.1 and Definition 3.2 that $f \in \mathcal{C}$ has support $[b - 1, c]$ if and only if the cdf G, such that $f(s) = G(s + 1) - G(s)$, has support $[b, c]$. Hence, f has support $[a, a + 1]$ if and only if G is degenerated at $a + 1$. A trivial calculation learns that in this case f is the right-continuous pdf of the uniform distribution on $[a, a + 1]$. $\square$

Corollary 3.3 *Let f be a pdf of the uniform distribution with support $[a, b]$, $a, b \in \mathbb{R}$. Then $f \in \mathcal{C}$ if and only if $b - a \in \mathbb{Z}_+$.*

PROOF. Consider the right-continuous version of f given by $f_+(s) = (b - a)^{-1}$ on $[a, b)$,

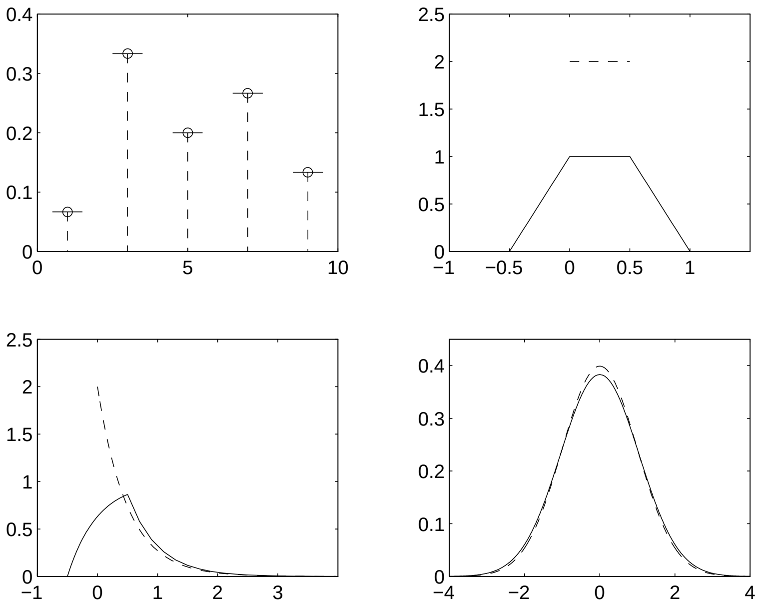


Figure 3.1: Illustrations to Example 3.1. Mass points are depicted by $\circ$*, the pdf of* F *by a dashed curve, and the pdf* f *by a solid curve. Top left: Part (b) for a discrete distribution on* $\{1, 3, 5, 7, 9\}$ *with probabilities* $1/15, 5/15, 3/15, 4/15, 2/15$*, respectively. Top right: Part (d). Bottom left: Part (e) with* $\lambda = 2$*. Bottom right: Part (f) with* $\mu = 0$ *and* $\sigma^2 = 1$*.*

and $f_+(s) = 0$ otherwise. By Corollary 3.1, the pdf f is in $\mathcal{C}$ if and only if $G(s) := \sum_{k=1}^{\infty} f_+(s-k)$, $s \in \mathbb{R}$, is a cdf with finite mean value. It holds

$$G(s) = (b-a)^{-1} N(s), \quad s \in \mathbb{R},$$

where $N(s) = |\{k \in \mathbb{Z}_+ \setminus \{0\} : s - k \in [a, b)\}|$. It remains to verify under which conditions on a and b the function G is a cdf.

Suppose $b - a = n + \varepsilon$ with $n \in \mathbb{Z}_+$ and $0 < \varepsilon < 1$. Then $N(a+1+n) = n+1$ so that $G(a+1+n) = (n+1)/(n+\varepsilon) > 1$. Obviously, G is not a cdf in this case. It remains to consider $b - a = n$ with $n \in \mathbb{Z}_+ \setminus \{0\}$. Then we get

$$G(s) = \begin{cases} 0, & s \in (-\infty, a+1), \\ k/n, & s \in [a+k, a+k+1), \quad k = 1, \ldots, n-1, \\ 1, & s \in [a+n, \infty), \end{cases}$$

and this is a cdf (with obviously a finite mean value), indeed. □

Finally, we give precise conditions such that a weighted sum (possibly non-denumer-able) of elements of $\mathcal{C}$ belongs to $\mathcal{C}$ itself.

Lemma 3.4 *Let, for any $t \in \mathbb{R}$, f_t be a pdf in $\mathcal{C}$ with first absolute moment $\mu(t)$. If Ψ is a cdf on $\mathbb{R}$ such that $\int \mu(t)\, d\Psi(t) < \infty$, then the function f, defined by*

$$f(s) := \int_{-\infty}^{\infty} f_t(s)\, d\Psi(t), \quad s \in \mathbb{R},$$

is a pdf in $\mathcal{C}$ too.

PROOF. We will show that f is generated by some cdf G with finite mean value as in Definition 3.2, so that f is a pdf in $\mathcal{C}$ by Theorem 3.1.

For any $t \in \mathbb{R}$, there exists a cdf G_t which generates the pdf f_t since $f_t \in \mathcal{C}$. Define

$$G(s) := \int_{-\infty}^{\infty} G_t(s)\, d\Psi(t), \quad s \in \mathbb{R}.$$

It is easy to see that $G(-\infty) = 0$, $G(\infty) = 1$ and that G is nondecreasing. It is right-continuous since, for $s \in \mathbb{R}$,

$$\lim_{u \downarrow s} G(u) = \lim_{u \downarrow s} \int_{-\infty}^{\infty} G_t(u)\, d\Psi(t) = \int_{-\infty}^{\infty} \lim_{u \downarrow s} G_t(u)\, d\Psi(t) = G(s),$$

where the second equality holds by Lebesgue's dominated convergence theorem (using that $G_t \le 1$). Hence G is a cdf. Moreover, following the proof of Theorem 3.1, G has a finite mean value since f does:

$$\int_{-\infty}^{\infty} |s| f(s)\, ds = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} |s| f_t(s)\, ds \right\} d\Psi(t) = \int_{-\infty}^{\infty} \mu(t)\, d\Psi(t) < \infty.$$

Finally, for $s \in \mathbb{R}$,

$$f(s) = \int_{-\infty}^{\infty} f_t(s)\, d\Psi(t) = \int_{-\infty}^{\infty} \big(G_t(s+1) - G_t(s)\big)\, d\Psi(t) = G(s+1) - G(s),$$

so that $f \in \mathcal{C}$. □

3.1 Reformulation in terms of random variables

Here we reformulate some of the results of the preceding section in terms of random variables. Note that the second part of the following lemma was already stated and proved in Lemma 3.3. However, the proof is more straightforward in this setting, and in particular it gives insight in the constant difference between the variances of a pdf $f \in \mathcal{C}$ and its generator cdf G.

Lemma 3.5 *Let G be an arbitrary cdf with finite mean value, and let η be a random variable distributed according to G. Let ξ be another random variable. Assume that, for each $s \in \mathbb{R}$, the conditional distribution of ξ given $\eta = s$ is the uniform distribution on $[s-1, s]$.*

(i) Then ξ is continuously distributed, and its pdf f is generated by G in the sense of Definition 3.2. In particular, $f \in \mathcal{C}$.

(ii) *Denote the mean value and variance of ξ and η by μ_f, σ_f^2 and μ_G, σ_G^2, respectively. Then $\mu_f = \mu_G - 1/2$ and $\sigma_f^2 = \sigma_G^2 + 1/12$.*

PROOF. (i) For $z \in \mathbb{R}$ it holds $\Pr\{\xi \le z\} = \int_{-\infty}^{\infty} \Pr\{\xi \le z | \eta = s\}\, dG(s)$. The assumption implies

$$\begin{aligned}
\Pr\{\xi \le z\} &= \int_{(-\infty,z]} dG(s) + \int_{(z,z+1]} (z+1-s)\, dG(s) \\
&= G(z) + \int_{(z,z+1]} \left\{ \int_s^{z+1} dt \right\} dG(s) \\
&= G(z) + \int_z^{z+1} \left\{ \int_{(z,t]} dG(s) \right\} dt \\
&= G(z) + \int_z^{z+1} \big(G(t) - G(z)\big)\, dt \\
&= \int_z^{z+1} G(t)\, dt.
\end{aligned}$$

With reference to Lemma 3.3, this completes the proof.

(ii) Denoting by $\mathbb{E}_\Phi[\cdot]$ the expectation with respect to the distribution with cdf Φ, we have

$$\mu_f = \mathbb{E}_G\left[\mathbb{E}_{F|G}[\xi|\eta]\right] = \mathbb{E}_G[\eta - 1/2] = \mu_G - 1/2$$

and

$$\sigma_f^2 = \mathrm{var}\left(\mathbb{E}_{F|G}[\xi|\eta]\right) + \mathbb{E}_G[\mathrm{var}\,\xi|\eta] = \mathrm{var}(\eta - 1/2) + 1/12 = \sigma_G^2 + 1/12.$$

□

This result implies the following alternative description of the set $\mathcal{C}$:

Corollary 3.4 *Let ξ be a random variable with pdf f. Then $f \in \mathcal{C}$ if and only if there exists a random variable η with finite mean value, such that for all $s \in \mathbb{R}$ the conditional distribution of ξ given $\eta = s$ is uniform on $[s-1, s]$.*

It is not difficult to see sufficiency of this condition. For example, convexity of the function $g(z) = \mathbb{E}_\xi\left[\lceil \xi - z\rceil^+\right]$, $z \in \mathbb{R}$, then follows from

$$\begin{aligned}
\mathbb{E}_\xi\left[\lceil \xi - z\rceil^+\right] &= \int_{-\infty}^{z} \lceil t - z\rceil f(t)\, dt \\
&= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{z} \lceil t - z\rceil \cdot 1_{[s-1,s]}(t)\, dt \right\} dG(s),
\end{aligned}$$

where G denotes the cdf of η, since the inner integral is already a convex function of z by Corollary 3.2.

On the other hand, since the condition is also necessary, we see that convexity of the functions g, h, and Q depends ultimately on the requirement that any conditional distribution of ξ be uniform on some interval of length 1. In this sense, the pdf of the uniform distribution with unit support is the nucleus of the set $\mathcal{C}$.

4. Continuous simple recourse representation of Q

Next we consider the representation of convex instances of Q as the one-dimensional expected value function $\bar{Q}$ of a continuous simple recourse problem. The latter function can be written as

$$\begin{aligned}\bar{Q}(z) &:= q^{+}\mathbb{E}_{\xi}\left[(\xi - z)^{+}\right] + q^{-}\mathbb{E}_{\xi}\left[(\xi - z)^{-}\right] \\ &= q^{+}\int_{z}^{\infty}\big(1 - F(t)\big)\,dt - q^{-}\int_{-\infty}^{z} F(t)\,dt,\end{aligned}$$

where F is the cdf of the random variable ξ, see e.g. [1] or [18]. Such a representation exists by Theorem 3.1 in [3].

Theorem 4.1 *Let ξ have a pdf $f \in \mathcal{C}$ and cdf F, and let G denote the cdf that generates f according to Definition 3.2. Then*

$$Q(z) = q^{+}\mathbb{E}_{\psi_G}\left[(\psi_G - z)^{+}\right] + q^{-}\mathbb{E}_{\psi_G}\left[(\psi_G - z)^{-}\right] + \frac{q^{+}q^{-}}{q^{+} + q^{-}}, \quad z \in \mathbb{R}, \tag{11}$$

where ψ_G is a random variable with cdf

$$W_G(s) = \frac{q^{+}}{q^{+} + q^{-}}G(s) + \frac{q^{-}}{q^{+} + q^{-}}G(s+1), \quad s \in \mathbb{R}. \tag{12}$$

PROOF. For $z \in \mathbb{R}$ we have, see (2) and (3),

$$\begin{aligned}Q(z) &= q^{+}\sum_{k=0}^{\infty}\big(1 - F(z+k)\big) - q^{-}\sum_{k=0}^{\infty}F(z-k) \\ &= q^{+}\int_{z}^{\infty}\big(1 - G(t)\big)\,dt - q^{-}\int_{-\infty}^{z}G(t+1)\,dt,\end{aligned} \tag{13}$$

where we used that $F(s) = \int_{s}^{s+1} G(t)\,dt$ by Lemma 3.3.

It follows that

$$Q'_{+}(z) = -q^{+}\big(1 - G(z)\big) + q^{-}G(z+1),$$

so that the cdf W_G follows trivially from Theorem 3.1 in [3] which states that $W_G(s) = (Q'_{+}(s) + q^{+})/(q^{+} + q^{-})$, $s \in \mathbb{R}$.

The constant in (11) is equal to

$$c = Q(z) - \left(q^+\mathbb{E}_{\psi_G}\left[(\psi_G - z)^+\right] + q^-\mathbb{E}_{\psi_G}\left[(\psi_G - z)^-\right]\right)$$

for an arbitrary value of $z \in \mathbb{R}$. Substitution of (13) gives

$$\begin{aligned}
c &= q^+\int_z^\infty \big(1 - G(t)\big)\, dt + q^-\int_{-\infty}^z G(t+1)\, dt \\
&\qquad - q^+\int_z^\infty \big(1 - W_G(t)\big)\, dt - q^-\int_{-\infty}^z W_G(t)\, dt \\
&= q^+\int_z^\infty \big(W_G(t) - G(t)\big)\, dt + q^-\int_{-\infty}^z \big(G(t+1) - W_G(t)\big)\, dt. \qquad (14)
\end{aligned}$$

Using (12) to substitute for W_G, we find that the integrands of the first and the second term in (14) are equal to $q^- f(t)/(q^+ + q^-)$ and $q^+ f(t)/(q^+ + q^-)$, respectively. The result now follows. □

We conclude that the function Q is equal (up to a constant) to the one-dimensional expected value function of a continuous simple recourse problem. In addition to its intrinsic theoretical value, this result is of practical use if the cdf G that generates the pdf f is known (actually, if we know G_i for all m_2 terms of the n_1-dimensional expected value function function $\mathcal{Q}$). In that case, we know all data of the continuous simple recourse model that is equivalent to our integer simple recourse model, and hence may solve the problem using existing special purpose software. The remainder of this paper is devoted to a class of approximations (of the underlying distribution, and hence of Q) for which it is possible to follow this approach.

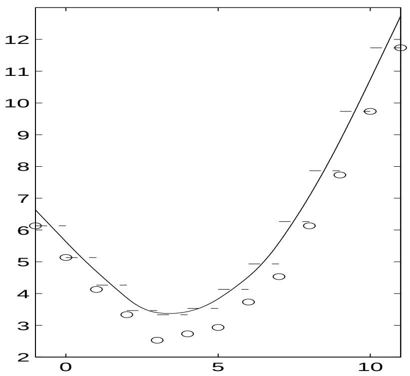
5. Definition of α-approximations

In Section 3 we defined the class $\mathcal{C}$ of probability density functions that correspond to convex expected value functions Q of simple integer recourse problems with fixed T matrix and random right-hand side parameters. If an arbitrary cdf F is approximated by a distribution in $\mathcal{C}$, the corresponding one-dimensional expected value function will be a convex approximation of Q. We will study here a subclass of $\mathcal{C}$ for approximating a cdf F, consisting of what we call *α-approximations*.

The definition of α-approximations below is independent of the exposition in the previous section. In Corollary 5.1 below it is indicated by which distribution the α-approximation is generated in the sense of the previous section.

Definition 5.1 For $\alpha \in [0, 1)$ and F a cdf of a random variable, the *α-approximation F_α* of F is defined as the piecewise linear function generated by the restriction of F to the lattice $\alpha + \mathbb{Z}$. That is, for any $\bar{s} \in \alpha + \mathbb{Z}$,

$$F_\alpha(s) := F(\bar{s}) + (s - \bar{s})\left(F(\bar{s}+1) - F(\bar{s})\right), \quad s \in [\bar{s}, \bar{s}+1].$$



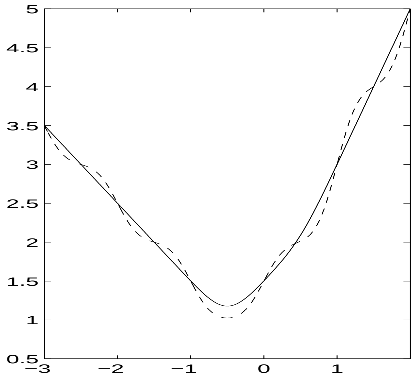


Figure 4.1: Approximations of the function Q ($q^+ = 1,\ q^- = 2$) by replacing the distribution F by the distribution $f(s) = F(s + 1/2) - F(s - 1/2)$, $s \in \mathbb{R}$. The function value in discontinuity points is depicted by $\circ$, the function Q by a dashed curve, and the approximation by a solid curve. Left: F is the discrete distribution on $\{1, 3, 5, 7, 9\}$ *with probabilities* $1/15, 5/15, 3/15, 4/15, 2/15$, *respectively. Right: F is the normal distribution with mean value* 0 *and variance* 0.05.

Since F_α is continuous and non-decreasing with $\lim_{s\to-\infty} F_\alpha(s) = 0$ and $\lim_{s\to\infty} F_\alpha(s) = 1$, it is a cdf itself. If ξ is a random variable with cdf F, any random variable ξ_α with cdf F_α will be called an α-approximation of ξ. That is, the distribution of any α-approximation ξ_α is characterized by the following two properties. For all $\bar{s} \in \alpha + \mathbb{Z}$,

(a) $\Pr\{\xi_\alpha \in (\bar{s}, \bar{s} + 1]\} = \Pr\{\xi \in (\bar{s}, \bar{s} + 1]\}$
(b) Given $\xi_\alpha \in (\bar{s}, \bar{s} + 1]$, its conditional distribution is uniform.

Notice that ξ_α has a pdf even if ξ does not.

Remark 5.1 In Definition 5.1 the α-approximation is based on the right continuous cdf F. An alternative definition of the α-approximation is based on the left continuous version $\hat{F}$ of F, in the following way. For any $\bar{s} \in \alpha + \mathbb{Z}$,

$$\hat{F}_\alpha(s) := \hat{F}(\bar{s}) + (s - \bar{s})\left(\hat{F}(\bar{s} + 1) - \hat{F}(\bar{s})\right), \quad s \in [\bar{s}, \bar{s} + 1].$$

Of course, if $\Pr\{\xi \in \alpha + \mathbb{Z}\} = 0$, both definitions lead to the same result. In particular this is true if ξ is continuously distributed. On the other hand, if $\Pr\{\xi \in \alpha + \mathbb{Z}\} > 0$ the alternative definition gives a distribution that is different from the one in Definition 5.1.

For the calculus with α-approximations we need the following generalization of the concepts of integer round down and integer round up.

Definition 5.2 For $s \in \mathbb{R}$ and $\alpha \in [0, 1)$, the *round down* and the *round up* of s *with respect* *to* $\alpha + \mathbb{Z}$ are defined as, respectively,

$$\begin{aligned} \lfloor s \rfloor_\alpha &:= \max\{\alpha + k : \alpha + k \le s,\ k \in \mathbb{Z}\} \\ \lceil s \rceil_\alpha &:= \min\{\alpha + k : \alpha + k \ge s,\ k \in \mathbb{Z}\}, \end{aligned}$$

i.e., $\lfloor s \rfloor_\alpha = \lfloor s - \alpha \rfloor + \alpha$ and $\lceil s \rceil_\alpha = \lceil s - \alpha \rceil + \alpha$.

Notice that for all $s \in \mathbb{R}$, $\alpha \in [0, 1)$, we have that $\lfloor s \rfloor_\alpha \le s \le \lceil s \rceil_\alpha$. In particular, if $s \in \alpha + \mathbb{Z}$ then $\lfloor s \rfloor_\alpha = s = \lceil s \rceil_\alpha$, and if $s \notin \alpha + \mathbb{Z}$ then $\lfloor s \rfloor_\alpha < s < \lceil s \rceil_\alpha = \lfloor s \rfloor_\alpha + 1$. Obviously, the usual integer round down and round up correspond to the case $\alpha = 0$.

The next lemma gives formulae for the α-approximation F_α of F and its pdf f_α that will be used frequently.

Lemma 5.1 *Let F be the cdf of a random variable. For $\alpha \in [0, 1)$, its α-approximation is the cdf F_α given by*

$$F_\alpha(s) = F(\lfloor s \rfloor_\alpha) + (s - \lfloor s \rfloor_\alpha)\left(F(\lceil s \rceil_\alpha) - F(\lfloor s \rfloor_\alpha)\right), \qquad s \in \mathbb{R}.$$

The right-continuous version f_α of the pdf of F_α is given by the piecewise constant function

$$f_\alpha(s) = F(\lfloor s \rfloor_\alpha + 1) - F(\lfloor s \rfloor_\alpha), \qquad s \in \mathbb{R}, \tag{15}$$

which is constant on every unit interval $[\alpha + k, \alpha + k + 1)$, $k \in \mathbb{Z}$.

In particular,

$$f_\alpha(s) = F(\lceil s \rceil_\alpha) - F(\lfloor s \rfloor_\alpha) = \int_{S_\alpha(s)} dF(t), \quad s \notin \alpha + \mathbb{Z},$$

where $S_\alpha(s) = (\lfloor s \rfloor_\alpha, \lceil s \rceil_\alpha]$.

PROOF. The formula for F_α is a reformulation of its definition for $s \notin \alpha + \mathbb{Z}$, but obviously it is also true for $s \in \alpha + \mathbb{Z}$. The formula for f_α follows from the one for F_α since f_α is the right derivative of F_α. $\square$

See Figure 5.1 for examples of α-approximations F_α and f_α.

We proceed by showing that $f_\alpha \in \mathcal{C}$ if it has a finite mean value. In this context, it is relevant to consider the distribution of the random variables $\lfloor \xi \rfloor_\alpha := \lfloor \xi - \alpha \rfloor + \alpha$ and $\lceil \xi \rceil_\alpha := \lceil \xi - \alpha \rceil + \alpha$ for a fixed $\alpha \in [0, 1)$, where ξ is a random variable with an arbitrary cdf F. Both $\lfloor \xi \rfloor_\alpha$ and $\lceil \xi \rceil_\alpha$ have values in $\alpha + \mathbb{Z}$, and

$$\begin{aligned} \Pr\{\lfloor \xi \rfloor_\alpha < s\} &= \Pr\{\xi < \lceil s \rceil_\alpha\} = \hat{F}(\lceil s \rceil_\alpha), \quad s \in \mathbb{R}, \\ \Pr\{\lceil \xi \rceil_\alpha \le s\} &= \Pr\{\xi \le \lfloor s \rfloor_\alpha\} = F(\lfloor s \rfloor_\alpha), \quad s \in \mathbb{R}, \end{aligned}$$

where we used that $\lfloor x \rfloor_\alpha < y \Leftrightarrow x < \lceil y \rceil_\alpha$ and $\lceil x \rceil_\alpha \le y \Leftrightarrow x \le \lfloor y \rfloor_\alpha$ for all $x, y \in \mathbb{R}$.

Corollary 5.1 *Let F be a cdf with finite mean value. Then $f_\alpha \in \mathcal{C}$ for all $\alpha \in [0, 1)$. In fact, f_α is generated by the cdf $G_\alpha(s) = F(\lfloor s \rfloor_\alpha)$, $s \in \mathbb{R}$ of $\lceil \xi \rceil_\alpha$, where ξ is a random variable with*

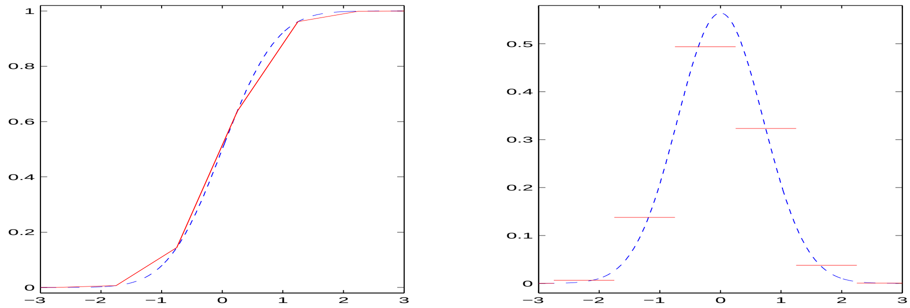


Figure 5.1: Let $\xi \sim \mathcal{N}(0, 1/2)$ and $\alpha = 1/4$. Left: cdfs F (dashed) and F_α. Right: pdfs f (dashed) and f_α.

cdf F.

PROOF. Immediate from (15) and Lemma 3.3. Obviously, G_α has finite mean value since F has. □

Remark 5.1 continued. It can be verified that the alternative definition of α-approxima-tions, based on the left-continuous version $\hat{F}$ of F, leads to the pdf generated by the distribution of $\lfloor \xi \rfloor_\alpha + 1$.

In the sequel we restrict ourselves to the case that ξ is continuously distributed. There are two reasons to accept this loss of generality. In the first place, for the case that ξ follows a discrete distribution we can resort to the convex hull of the function Q, see [3, 4]. Because of the favorable properties of the convex hull, there is no need to investigate other approximations for this class. Secondly, the analysis of the general case is more complicated, since it appears that for the α-approximation of the expected shortage function h the alternative definition in Remark 5.1 is appropriate, whereas Definition 5.1 is suitable in case of the expected surplus function g. Consequently, for discretely distributed ξ it is quite possible that different α-approximations are needed in the two constituents of the function Q. So far we have only been able to prove interesting results for continuously distributed ξ.

6. Analysis of α-approximations

We first study α-approximations of the expected surplus function g. We derive a bound on the error of the approximations that is uniform in $\alpha \in [0, 1)$. In Subsection 6.2 we consider convex combinations of α-approximations. Subsection 6.3 contains the corresponding results for the expected shortage function h. In Subsection 6.4 we first combine the preceding results to obtain convex approximations Q_α for the one-dimensional expected value function Q, and then discuss interpretations of the above-mentioned uniform error bound. In Section 7 we consider the representation of Q_α as the one-dimensional expected value function of a continuous simple recourse problem. In particular, it will be shown that the distribution of the random variable appearing in the latter problem can be calculated directly. Subsequently, in Section 8 we extend the results to the n_1-dimensional expected value function $\mathcal{Q}$, and discuss the relation between the optimal values of a simple integer recourse problem and its α-approximations.

6.1 The expected surplus function

In the following lemma we define α-approximations of the expected surplus function g.

Lemma 6.1 *Let ξ be a random variable with $\mathbb{E}_\xi[\xi] \in \mathbb{R}$. For all $\alpha \in [0, 1)$, let ξ_α be an α-approximation of ξ. For each $\alpha \in [0, 1)$ define*

$$g_\alpha(z) := \mathbb{E}_{\xi_\alpha}\left[\lceil \xi_\alpha - z \rceil^+\right], \qquad z \in \mathbb{R}.$$

Then $g_\alpha(z) = \sum_{k=0}^\infty (1 - F_\alpha(z+k))$, $z \in \mathbb{R}$. For all $\alpha \in [0, 1)$ it is a convex function.

The function g_α is completely determined by the expected surplus function $g(z) = \mathbb{E}_\xi\left[\lceil \xi - z \rceil^+\right]$ in the following way. It is the piecewise linear function generated by the restriction of g to $\alpha+\mathbb{Z}$. That is,

$$\begin{aligned} g_\alpha(z) &= g(\lfloor z \rfloor_\alpha) + (z - \lfloor z \rfloor_\alpha)\left(g(\lceil z \rceil_\alpha) - g(\lfloor z \rfloor_\alpha)\right) \\ &= g(\lfloor z \rfloor_\alpha) + (z - \lfloor z \rfloor_\alpha)\left(F(\lfloor z \rfloor_\alpha) - 1\right), \quad z \in \mathbb{R}. \end{aligned}$$

In particular, $g_\alpha(z) = g(z)$ for $z \in \alpha + \mathbb{Z}$.

PROOF. Since f_α, the pdf of ξ_α, is in $\mathcal{C}$ by Corollary 5.1, we know that g_α is a convex function satisfying $g_\alpha(z) = \sum_{k=0}^\infty (1 - F_\alpha(z+k))$, $z \in \mathbb{R}$. Similarly, $g(z) = \sum_{k=0}^\infty (1 - F(z+k))$. Restricting the attention to $z \in \alpha + \mathbb{Z}$, we see that $z + k \in \alpha + \mathbb{Z}$ for all $k \in \mathbb{Z}$, so that $F_\alpha(z+k) = F(z+k)$ for all $k \in \mathbb{Z}$. Consequently, g_α coincides with g on $\alpha+\mathbb{Z}$. Consider next the case $z \in [\bar{z}, \bar{z}+1]$ for any $\bar{z} \in \alpha+\mathbb{Z}$. Then for each $k \in \mathbb{Z}$ we have $z+k \in [\bar{z}+k, \bar{z}+k+1]$ with $\bar{z}+k$ and $\bar{z}+k+1$ two neighboring points in $\alpha+\mathbb{Z}$. Hence, $1 - F_\alpha(z+k)$ is (affine-)linear in z on $[\bar{z}, \bar{z}+1]$ for each $k \in \mathbb{Z}$, and so is $g_\alpha(z) = \sum_{k=0}^\infty (1 - F_\alpha(z+k))$. Consequently, for $\bar{z} \in \alpha + \mathbb{Z}$,

$$\begin{aligned} g_\alpha(z) &= g_\alpha(\bar{z}) + (z - \bar{z})\left(g_\alpha(\bar{z}+1) - g_\alpha(\bar{z})\right) \\ &= g(\bar{z}) + (z - \bar{z})\left(g(\bar{z}+1) - g(\bar{z})\right), \quad z \in [\bar{z}, \bar{z}+1]. \end{aligned}$$

Since, for $z \in \mathbb{R}$, $\lfloor z \rfloor_\alpha \le z \le \lceil z \rceil_\alpha$, $\lfloor z \rfloor_\alpha \in \alpha + \mathbb{Z}$, and $\lceil z \rceil_\alpha = \lfloor z \rfloor_\alpha + 1$ unless $z - \lfloor z \rfloor_\alpha = 0$, this implies

$$\begin{aligned} g_\alpha(z) &= g(\lfloor z \rfloor_\alpha) + (z - \lfloor z \rfloor_\alpha)\left(g(\lceil z \rceil_\alpha) - g(\lfloor z \rfloor_\alpha)\right) \\ &= g(\lfloor z \rfloor_\alpha) + (z - \lfloor z \rfloor_\alpha)\left(F(\lfloor z \rfloor_\alpha) - 1\right), \quad z \in \mathbb{R}, \end{aligned}$$

where the last equality follows directly from (2). □

Remark 6.1 Instead of using Corollary 5.1 to prove that g_α is convex, this can also be seen directly from the relation between the functions g_α and g. Convexity follows immediately from the fact that the cdf F is non-decreasing. Alternatively, since g_α is piecewise linear and coincides with g on the set $\alpha + \mathbb{Z}$, it is convex by Lemma 3.1.

In the following we assume that the random variable ξ in the definition of the expected surplus function g is continuously distributed with pdf $f \in \mathcal{F}$. Under this assumption we will derive a bound, independent of α, on the uniform distance between g_α and g: $\|g_\alpha - g\|_\infty = \sup_{z\in\mathbb{R}} |g_\alpha(z) - g(z)|$. The supremum norm is the appropriate norm here since it will later provide directly a bound on the difference between the minimum value of $\mathcal{Q}$ and that of its α-approximation $\mathcal{Q}_\alpha$ (see Section 8).

Remark 6.2 Independent of the distribution type of ξ we have the trivial error bound $\|g_\alpha - g\|_\infty \le 1$. This bound follows directly from the monotonicity of g_α and g and the observation that, for all $z \in \mathbb{R}$, both $g_\alpha(z)$ and $g(z)$ are bounded by $g(\lfloor z \rfloor_\alpha)$ and $g(\lfloor z \rfloor_\alpha + 1)$. Hence

$$|g_\alpha(z) - g(z)| \le g(\lfloor z \rfloor_\alpha) - g(\lfloor z \rfloor_\alpha + 1) = 1 - F(\lfloor z \rfloor_\alpha) \le 1 \qquad \forall z \in \mathbb{R},$$

where F is the cdf of ξ.

Theorem 6.1 *Assume that ξ has a pdf $f \in \mathcal{F}$. Then, for all $z \in \mathbb{R}$ and all $\alpha \in [0, 1)$,*

$$|g_\alpha(z) - g(z)| \le \min\left\{z - \lfloor z \rfloor_\alpha, \lceil z \rceil_\alpha - z\right\} \frac{|\Delta| f}{2},$$

so that

$$\|g_\alpha - g\|_\infty \le \frac{|\Delta| f}{4},$$

where $|\Delta| f$ denotes the total variation of f on $\mathbb{R}$.

In particular, if the pdf f is unimodal then $\|g_\alpha - g\|_\infty \le f(\nu)/2$ for all $\alpha \in [0, 1)$, where ν is the mode of the distribution.

The rather technical proof of Theorem 6.1 is presented in Appendix 9.2.

In many cases, the error bound given in Theorem 6.1 is sharper than the trivial bound $\|g_\alpha - g\|_\infty \le 1$. Only if $|\Delta| f > 4$, which for example for normal distributions corresponds to a variance less than 0.04, this is not the case.

In Section 6.4 where we present similar results for the function Q, we will discuss interpretations of the error bound.

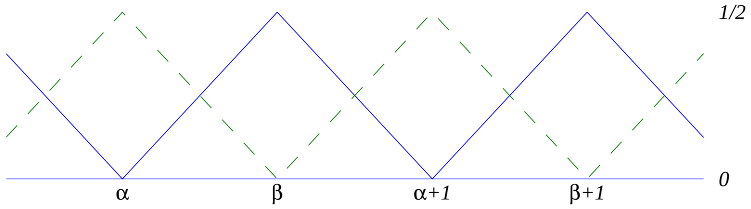


Figure 6.1: The functions m_α and m_β (dashed) with $\beta - \alpha = 1/2$.

6.2 Convex combinations of α-approximations

For each $\alpha \in [0, 1)$ we defined the α-approximation F_α of the cdf F of ξ, and derived a uniform upper bound on the difference of the corresponding expected surplus functions g_α and g in case F has a pdf f. Moreover, we showed that F_α has a pdf f_α, and if F has a finite mean value then $f_\alpha \in \mathcal{C}$. We now consider convex combinations of α-approximations $\sum_{i=1}^n \lambda_i F_{\alpha_i}$, with $\lambda_i > 0$, $\sum_{i=1}^n \lambda_i = 1$, $\alpha_i \in [0, 1)$. Due to Lemma 3.4, such distributions yield convex expected surplus functions too. We will show that by taking convex combinations the uniform upper bound can be reduced by a factor 2.

First we consider a special case. For any $\alpha, \beta \in [0, 1)$, we define

$$f_{\alpha\beta}(s) = \frac{f_\alpha(s) + f_\beta(s)}{2}, \qquad s \in \mathbb{R},$$

where f_γ, $\gamma = \alpha, \beta$, is the right-continuous version of the pdf of the γ-approximation of F.

To facilitate the exposition below, we introduce the following notation. For $\alpha \in [0, 1)$, define

$$m_\alpha(z) := \min\left\{z - \lfloor z \rfloor_\alpha, \lceil z \rceil_\alpha - z\right\}, \qquad z \in \mathbb{R}.$$

The function m_α is depicted in Figure 6.1. First of all, we note that $m_\alpha(z) \leq 1/2$ for all $z \in \mathbb{R}$ and all $\alpha \in [0, 1)$. Moreover, for all $z \in \mathbb{R}$

$$m_\alpha(z) + m_\alpha(z + 1/2) = 1/2 \qquad \forall \alpha \in [0, 1). \tag{16}$$

This relation is easy to prove for the special case $\alpha = 0$, and the general case then follows from $m_\alpha(z) = m_0(z - \alpha)$, $z \in \mathbb{R}$, which is easily derived using Definition 5.2. Finally, again using Definition 5.2 we see that for all $\alpha \in [0, 1/2)$ it holds

$$\lfloor z + 1/2 \rfloor_\alpha = \lfloor z - 1/2 - \alpha \rfloor + \alpha + 1 = \lfloor z \rfloor_{\alpha + \frac{1}{2}} + 1/2 \qquad \forall z \in \mathbb{R}.$$

It follows that if $|\alpha - \beta| = 1/2$ then

$$m_\alpha(z + 1/2) = m_\beta(z) \qquad \forall z \in \mathbb{R}. \tag{17}$$

Corollary 6.1 *Let F be a cdf with finite mean value. For $\alpha, \beta \in [0, 1)$, let $\xi_{\alpha\beta}$ be any random* *variable with pdf $f_{\alpha\beta}$. Then the corresponding expected surplus function $g_{\alpha\beta}$, defined by*

$$g_{\alpha\beta}(z) = \mathbb{E}_{\xi_{\alpha\beta}}\left[\lceil \xi_{\alpha\beta} - z\rceil^{+}\right], \qquad z \in \mathbb{R},$$

is piecewise linear and convex. Moreover, if F has a pdf f with $f \in \mathcal{F}$ and if $|\alpha - \beta| = 1/2$, then

$$\|g_{\alpha\beta} - g\|_{\infty} \leq \frac{|\Delta| f}{8}. \tag{18}$$

PROOF. It is easy to see that

$$g_{\alpha\beta}(z) = \frac{g_{\alpha}(z) + g_{\beta}(z)}{2}, \quad z \in \mathbb{R},$$

so that $g_{\alpha\beta}$ is piecewise linear and convex by Lemma 6.1.

To prove the second statement, we use that for all $z \in \mathbb{R}$

$$\begin{aligned} \left|g_{\alpha\beta}(z) - g(z)\right| &\leq \frac{|g_{\alpha}(z) - g(z)| + \left|g_{\beta}(z) - g(z)\right|}{2} \\ &\leq \left(m_{\alpha}(z) + m_{\beta}(z)\right) \frac{|\Delta| f}{4} \end{aligned}$$

where the second inequality follows from Theorem 6.1.

By our choice of α and β it follows from (16) and (17) that $m_{\alpha}(z) + m_{\beta}(z) \equiv 1/2$, which is illustrated in Figure 6.1. This completes the proof. □

Given Corollary 6.1, one might expect further improvement of the error bound (18) if the approximation is based on combinations of more than two piecewise constant pdfs f_{α}, $\alpha \in [0, 1)$. However, we will prove that this is not the case.

Since we wish to consider all possible convex combinations of pdfs f_{α}, $\alpha \in [0, 1)$, it is convenient to interpret α as a random variable with support contained in $[0, 1)$. To avoid possible confusion, we will write $\tilde{\alpha}$ to denote the random variable whereas α is a scalar in $[0, 1)$ as before.

Theorem 6.2 *Let $\mathcal{A}$ denote the collection of all cdfs of distributions on the half-open interval* $[0, 1)$. *Then*

$$\min_{\Phi \in \mathcal{A}} \max_{z \in \mathbb{R}} \int m_s(z)\, d\Phi(s) = \frac{1}{4}.$$

PROOF. For any $\Phi \in \mathcal{A}$

$$\begin{aligned} &\sup_{z \in \mathbb{R}} \int m_s(z)\, d\Phi(s) \\ &= \sup_{z \in \mathbb{R}} \left\{ \max \left\{ \int m_s(z)\, d\Phi(s), \int m_s(z + 1/2)\, d\Phi(s) \right\} \right\} \end{aligned}$$

$$
\begin{aligned}
&\geq \sup_{z\in\mathbb{R}} \left\{ \frac{1}{2} \left(\int m_s(z)\, d\Phi(s) + \int m_s(z+1/2)\, d\Phi(s) \right) \right\} \\
&= \sup_{z\in\mathbb{R}} \int \frac{1}{2} \big(m_s(z) + m_s(z+1/2) \big)\, d\Phi(s) = \frac{1}{4},
\end{aligned}
$$

where the final equality follows from (16). The result now follows, since Corollary 6.1 implies that if the random variable $\tilde{\alpha}$ has support $\{\alpha, \beta\}$ such that $|\alpha - \beta| = 1/2$ and $\Pr\{\tilde{\alpha} = \alpha\} = \Pr\{\tilde{\alpha} = \beta\} = 1/2$ then

$$
\int m_s(z)\, d\Phi_{\tilde{\alpha}}(s) \quad = \quad \frac{1}{2} \big(m_s(z) + m_s(z+1/2) \big) = \frac{1}{4} \qquad \forall z \in \mathbb{R},
$$

where $\Phi_{\tilde{\alpha}}$ denotes the cdf of $\tilde{\alpha}$. □

We conclude that, compared to the error bound given in Theorem 6.1, the error bound can be reduced by a factor 2 if we allow convex combinations of pdfs f_α. However, in that case we lose the property that the approximation coincides with g at all points in $\alpha + \mathbb{Z}$ for some $\alpha \in [0, 1)$.

6.3 The expected shortage function

Next we turn to approximations of the expected shortage function $h(z) = \mathbb{E}_\xi \left[\lfloor \xi - z \rfloor^- \right]$, $z \in \mathbb{R}$. Recall that by Lemma 2.1 it holds that $h(z) = g^\zeta(-z)$ for all z, where g^ζ is the function that is obtained from g if the random variable ξ is replaced by $\zeta = -\xi$. Moreover, ζ has cdf $F_\zeta(s) = 1 - \hat{F}(-s)$, $s \in \mathbb{R}$, where $\hat{F}$ is the left continuous version of F. Since we assume that ξ is continuously distributed, we have $F_\zeta(s) = 1 - F(-s)$, $s \in \mathbb{R}$. If ξ has pdf f then ζ has pdf $f_\zeta(s) = f(-s)$, $s \in \mathbb{R}$, with total variation $|\Delta| f_\zeta = |\Delta| f$.

Consequently, all results derived for the approximations g_α trivially imply corresponding results for the approximations $h_\alpha(z) = \mathbb{E}_{\xi_\alpha} \left[\lfloor \xi_\alpha - z \rfloor^- \right]$, $z \in \mathbb{R}$, $\alpha \in [0, 1)$. Instead of listing them separately, we refer to the results below for the function Q_α, which on choosing $q^+ = 0$ and $q^- = 1$ apply to the function h_α.

Remark 6.3 Note that the indicated results for the function h are based on the fact that $F_\alpha(s) = F(s) = \hat{F}(s)$, $s \in \alpha + \mathbb{Z}$, which is true by the assumption that ξ is a continuous random variable (see Remark 5.1). As an example of the problems that arise if ξ is discretely distributed, we note that instead of $h_\alpha(z) = h(z)$ for all $z \in \alpha + \mathbb{Z}$, we obtain $h_\alpha(z) = h(z) + \Pr\{\xi \in z - \mathbb{Z}_+\} = \lim_{s \downarrow z} h(s)$, for all such z.

6.4 The one-dimensional expected value function

By combining the results for g_α and h_α we obtain the corresponding results for the α-approximations $Q_\alpha = q^+ g_\alpha + q^- h_\alpha$, $\alpha \in [0, 1)$, of the one-dimensional expected value function Q.

Theorem 6.3 *Let ξ be a continuous random variable with pdf f. For all $\alpha \in [0, 1)$, let ξ_α*

be an α-approximation of ξ. For each $\alpha \in [0, 1)$ define

$$Q_\alpha(z) := q^+\mathbb{E}_{\xi_\alpha}\left[\lceil \xi_\alpha - z\rceil^+\right] + q^-\mathbb{E}_{\xi_\alpha}\left[\lfloor \xi_\alpha - z\rfloor^-\right], \qquad z \in \mathbb{R},$$

where q^+ and q^- are non-negative scalars. Then

(a) *For each α, the function Q_α is convex. It is related to the function $Q(z) = q^+\mathbb{E}_\xi[\lceil \xi - z\rceil^+] + q^-\mathbb{E}_\xi\left[\lfloor \xi - z\rfloor^-\right]$ by the equation*

$$\begin{aligned} Q_\alpha(z) &= Q(\lfloor z\rfloor_\alpha) + (z - \lfloor z\rfloor_\alpha)\left(Q(\lceil z\rceil_\alpha) - Q(\lfloor z\rfloor_\alpha)\right) \\ &= Q(\lfloor z\rfloor_\alpha) + (z - \lfloor z\rfloor_\alpha)\left(-q^+ + q^+F(\lfloor z\rfloor_\alpha) + q^-F(\lceil z\rceil_\alpha)\right), \end{aligned}$$

so that Q_α is the piecewise linear function that coincides with Q at all points $\alpha + k$, $k \in \mathbb{Z}$.

(b) *Independent of the distribution type of ξ it holds $\|Q_\alpha - Q\|_\infty \le 1$.*
Assume that the pdf $f \in \mathcal{F}$. Then, for all $z \in \mathbb{R}$,

$$|Q_\alpha(z) - Q(z)| \le \min\left\{z - \lfloor z\rfloor_\alpha, \lceil z\rceil_\alpha - z\right\}(q^+ + q^-)\frac{|\Delta|f}{2},$$

so that

$$\|Q_\alpha - Q\|_\infty \le (q^+ + q^-)\frac{|\Delta|f}{4}.$$

In particular, if the pdf f is unimodal then

$$\|Q_\alpha - Q\|_\infty \le (q^+ + q^-)\frac{f(\nu)}{2},$$

where ν is the mode of the distribution.

(c) *Let, for α and β different numbers in $[0, 1)$, $\xi_{\alpha\beta}$ be a random variable that is equal to ξ_α with probability $1/2$ and to ξ_β with the complementary probability. It has a piecewise constant right-continuous pdf $f_{\alpha\beta}(s) = \left(f_\alpha(s) + f_\beta(s)\right)/2$, $s \in \mathbb{R}$. Then*

$$Q_{\alpha\beta}(z) = q^+\mathbb{E}_{\xi_{\alpha\beta}}\left[\lceil \xi_{\alpha\beta} - z\rceil^+\right] + q^-\mathbb{E}_{\xi_{\alpha\beta}}\left[\lfloor \xi_{\alpha\beta} - z\rfloor^-\right], \qquad z \in \mathbb{R},$$

is a piecewise linear convex function. If $f \in \mathcal{F}$ and in addition it holds $|\alpha - \beta| = 1/2$, then

$$\|Q_{\alpha\beta} - Q\|_\infty \le (q^+ + q^-)\frac{|\Delta|f}{8}.$$

Moreover, this uniform error bound can not be reduced by using other convex combinations of pdfs of type f_α.

See Figure 6.2 for an example of the functions Q and Q_α.

Now it is time to reflect on the interpretation of the error bound that we have established for the case that ξ follows a continuous distribution. By Theorem 6.3 the error bound is proportional to the total variation of the pdf f of ξ. Computational experiments suggest that for many distributions the total variation of a pdf decreases as the variance of the distribution increases. For example, this correspondence holds if such a change of the distribution means that the probability mass is spread out more evenly over the (perhaps wider) support. Consequently, we would expect that the approximation Q_α becomes better as the variance in the distribution of ξ becomes higher. This expectation is supported by many examples, including the following.

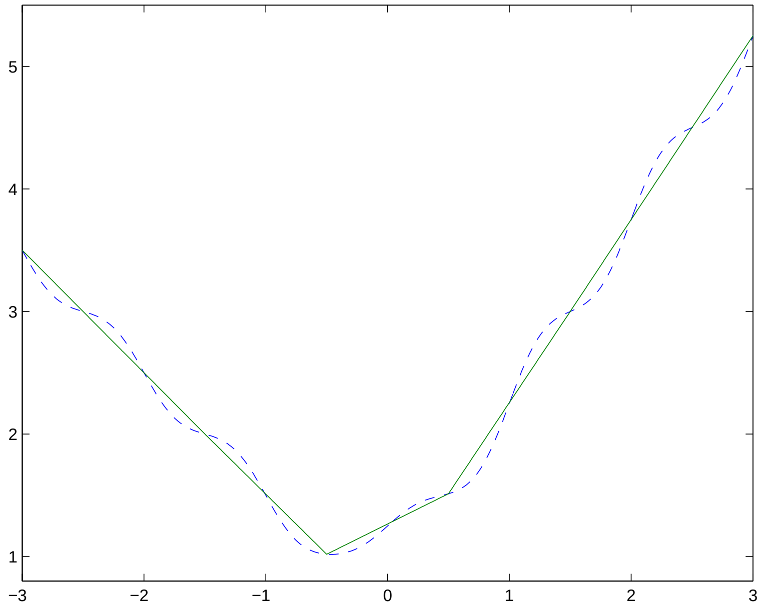


Figure 6.2: The functions Q (dashed) and Q_α in case $\xi \sim \mathcal{N}(0, 0.05)$, $q^+ = 1$, $q^- = 1.5$, and $\alpha = 0.5$.

Example 6.1 Assume that $\xi \sim \mathcal{N}(\mu, \sigma^2)$ and that $q^+ = q^- = 1$. Then, for all $z \in \mathbb{R}$ and every $\alpha \in [0, 1)$,

$$\begin{aligned} \|Q_\alpha - Q\|_\infty &\leq (q^+ + q^-)\frac{|\Delta|f}{4} \\ &= f(\mu) = \frac{1}{\sqrt{2\pi}\sigma}, \end{aligned}$$

where we used that the pdf of the normal distribution is unimodal with mode μ.

$\triangleleft$

Another interpretation of the error bound comes to mind. Let $\mathcal{G}$ denote the collection of all finite convex functions on $\mathbb{R}$. Then we may measure the *degree of non-convexity* of any finite function $\varphi : \mathbb{R} \mapsto \mathbb{R}$ by its distance $d(\varphi \,|\, \mathcal{G})$ to this set, defined as

$$d(\varphi \,|\, \mathcal{G}) := \inf_{v \in \mathcal{G}} \|\varphi - v\|_\infty.$$

We see that $d(\varphi \,|\, \mathcal{G}) = 0$ if a finite function φ is convex, whereas $d(\varphi \,|\, \mathcal{G}) > 0$ if φ is non-convex. From Theorem 6.3 we conclude that the degree of non-convexity of Q is bounded by a multiple of the total variation of the pdf f:

$$\begin{aligned} d(Q \mid \mathcal{G}) &= \inf_{v \in \mathcal{G}} \|Q - v\|_\infty \\ &\leq \|Q - Q_{\alpha\beta}\|_\infty \\ &\leq (q^+ + q^-)\frac{|\Delta| f}{8}. \end{aligned}$$

Since this upper bound can not be reduced by using other convex combinations of densities of the type f_α that are known to generate convex functions, we conjecture that this upper bound is rather sharp. This means that the degree of non-convexity of Q decreases if the total variation of the underlying pdf f decreases.

7. Continuous simple recourse representation of Q_α

Next we consider the representation of Q_α as the one-dimensional expected value function of a continuous simple recourse problem, as implied by Theorem 4.1.

Corollary 7.1 *Let ξ be a continuous random variable with cdf F with finite mean value, and $\alpha \in [0, 1)$. Denote, as before, the α-approximation of the one-dimensional expected value function by Q_α. Then*

$$Q_\alpha(z) = q^+ \mathbb{E}_{\psi_\alpha}\left[(\psi_\alpha - z)^+\right] + q^- \mathbb{E}_{\psi_\alpha}\left[(\psi_\alpha - z)^-\right] + \frac{q^+ q^-}{q^+ + q^-}, \quad z \in \mathbb{R}, \tag{19}$$

where ψ_α is a random variable with cdf

$$W_\alpha(s) = \frac{q^+}{q^+ + q^-} F(\lfloor s \rfloor_\alpha) + \frac{q^-}{q^+ + q^-} F(\lfloor s \rfloor_\alpha + 1), \quad s \in \mathbb{R}. \tag{20}$$

That is, ψ_α is a discrete random variable with support in $\alpha + \mathbb{Z}$ and

$$\begin{aligned} \Pr\{\psi_\alpha = \alpha + k\} &= \frac{q^+}{q^+ + q^-}\big(F(\alpha + k) - F(\alpha + k - 1)\big) \\ &\quad + \frac{q^-}{q^+ + q^-}\big(F(\alpha + k + 1) - F(\alpha + k)\big), \quad k \in \mathbb{Z}. \end{aligned}$$

PROOF. By Corollary 5.1, $f_\alpha \in \mathcal{C}$ since it is generated by $F(\lfloor \cdot \rfloor_\alpha)$. Therefore, the equations (19) and (20) follow from Theorem 4.1, which proves these results for any $f \in \mathcal{C}$. □

We conclude that the function Q_α is equal (up to a constant) to the one-dimensional expected value function of a continuous simple recourse problem, and that *the discrete distribution of its right-hand side random variable ψ_α can be computed directly from the distribution of ξ.*

The following corollary is an alternative formulation of the latter result in terms of random variables. It characterizes ψ_α as a 'probabilistic' round of ξ with respect to $\alpha + \mathbb{Z}$:

$$\Pr\{\psi_\alpha = \lfloor \xi \rfloor_\alpha\} = \frac{q^-}{q^+ + q^-}, \qquad \Pr\{\psi_\alpha = \lceil \xi \rceil_\alpha\} = \frac{q^+}{q^+ + q^-}.$$

Corollary 7.2 *Assume the setting of Corollary 7.1. Define the discrete random variable η, independent of ξ, such that*

$$\Pr\{\eta = 0\} = \frac{q^+}{q^+ + q^-}, \qquad \Pr\{\eta = 1\} = \frac{q^-}{q^+ + q^-}.$$

Then $\lceil \xi \rceil_\alpha - \eta$ has the same distribution as ψ_α.

PROOF. Obviously, the support of ψ_α is a subset of $\alpha + \mathbb{Z}$. Defining $p_0 := q^+/(q^+ + q^-)$ and $p_1 := q^-/(q^+ + q^-)$, it holds, for $k \in \mathbb{Z}$,

$$\begin{aligned}
&\Pr\{\lceil \xi \rceil_\alpha - \eta = \alpha + k\} \\
&= \Pr\{\lceil \xi \rceil_\alpha = \alpha + k \wedge \eta = 0\} + \Pr\{\lceil \xi \rceil_\alpha = \alpha + k + 1 \wedge \eta = 1\} \\
&= p_0 \Pr\{\alpha + k - 1 < \xi \le \alpha + k\} + p_1 \Pr\{\alpha + k < \xi \le \alpha + k + 1\} \\
&= p_0 \big(F(\alpha + k) - F(\alpha + k - 1)\big) + p_1 \big(F(\alpha + k + 1) - F(\alpha + k)\big) \\
&= \Pr\{\psi_\alpha = \alpha + k\}.
\end{aligned}$$

□

The function $Q_{\alpha\beta}$, defined in Theorem 6.3 (c), is also representable as a one-dimen-sional continuous recourse expected value function.

Corollary 7.3 *Let ξ be a continuous random variable with cdf F. Then the one-dimen-sional expected value function corresponding to $\xi_{\alpha\beta}$ can be written as follows.*

$$Q_{\alpha\beta}(z) = q^+ \mathbb{E}_{\psi_{\alpha\beta}}\left[(\psi_{\alpha\beta} - z)^+\right] + q^- \mathbb{E}_{\psi_{\alpha\beta}}\left[(\psi_{\alpha\beta} - z)^-\right] + \frac{q^+ q^-}{q^+ + q^-}, \quad z \in \mathbb{R},$$

where $\psi_{\alpha\beta}$ is a random variable with cdf

$$\begin{aligned}
W_{\alpha\beta}(s) &= \frac{q^+}{q^+ + q^-} \frac{F(\lfloor s \rfloor_\alpha) + F(\lfloor s \rfloor_\beta)}{2} \\
&\quad + \frac{q^-}{q^+ + q^-} \frac{F(\lfloor s \rfloor_\alpha + 1) + F(\lfloor s \rfloor_\beta + 1)}{2}, \quad s \in \mathbb{R}.
\end{aligned}$$

Hence we may use the following transformation to get $\psi_{\alpha\beta}$ from ξ:

$$\psi_{\alpha\beta} = \begin{cases} \lceil \xi \rceil_\alpha \text{ w.p. } p_0/2; \\ \lfloor \xi \rfloor_\alpha \text{ w.p. } p_1/2; \\ \lceil \xi \rceil_\beta \text{ w.p. } p_0/2; \\ \lfloor \xi \rfloor_\beta \text{ w.p. } p_1/2, \end{cases}$$

where $p_0 := q^+/(q^+ + q^-)$ *and* $p_1 := q^-/(q^+ + q^-)$.

PROOF. It follows from the definition of $Q_{\alpha\beta}$ that it is equal to $(Q_\alpha + Q_\beta)/2$. Hence, by Corollary 7.1 we have $W_{\alpha\beta} = (W_\alpha + W_\beta)/2$, which is the formula given above.

The relation between $\psi_{\alpha\beta}$ and ξ is proved in the same way as the corresponding result in Corollary 7.2. □

8. The expected value function

Here we consider α-approximations, similar to the ones explained in the previous sections, of the n_1-dimensional expected value function $\mathcal{Q}$, which we recall is given by

$$\mathcal{Q}(x) := \sum_{i=1}^{m_2} Q_i(T_i x), \qquad x \in \mathbb{R}^{n_1},$$

where

$$Q_i(T_i x) := q_i^+ \mathbb{E}_{\xi_i}\left[\lceil \xi_i - T_i x\rceil^+\right] + q_i^- \mathbb{E}_{\xi_i}\left[\lfloor \xi_i - T_i x\rfloor^-\right],$$

T_i is the ith row of the technology matrix matrix T, and ξ_i is the ith component of the m_2-dimensional random vector ξ.

Below we extend the results of the previous sections to the higher-dimensional case. All results follow directly from the one-dimensional case, and therefore we omit the proof. Note that below α denotes an m_2-dimensional vector in $[0,1)^{m_2}$; each of its components α_i corresponds to the function Q_i, $i = 1, \dots, m_2$. To alleviate notational burden, we use e.g. Q_{α_i} instead of Q_{i,α_i} to denote the α_i-approximation of Q_i; analogously, $f_{\alpha\beta_i}$ denotes the $\alpha_i\beta_i$-approximation of a pdf f_i.

Theorem 8.1 *Let $\xi = (\xi_1, \dots, \xi_{m_2})$ be a random vector with finite mean value, and assume that each component is continuously distributed. Let f_i denote the marginal pdf of ξ_i, $i = 1, \dots, m_2$. For each $\alpha = (\alpha_1, \dots, \alpha_{m_2}) \in [0,1)^{m_2}$, let ξ_{α_i}, $i = 1, \dots, m_2$, be an α-approximation of ξ_i. For each $\alpha \in [0,1)^{m_2}$ define the α-approximation of the expected value function $\mathcal{Q}$ as*

$$\mathcal{Q}_\alpha(x) := \sum_{i=1}^{m_2} Q_{\alpha_i}(T_i x), \qquad x \in \mathbb{R}^{n_1},$$

where

$$Q_{\alpha_i}(T_i x) := q_i^+ \mathbb{E}_{\xi_{\alpha_i}}\left[\lceil \xi_{\alpha_i} - T_i x\rceil^+\right] + q_i^- \mathbb{E}_{\xi_{\alpha_i}}\left[\lfloor \xi_{\alpha_i} - T_i x\rfloor^-\right].$$

Then

(a) *For each α, the function $\mathcal{Q}_\alpha$ is convex and polyhedral. It is affine on every polyhedral set*

$$\left\{x \in \mathbb{R}^{n_1} : k + \alpha \le Tx \le k + \mathbf{1} + \alpha\right\}, \qquad k \in \mathbb{Z}^{m_2},$$

where $\mathbf{1}$ denotes the m_2-dimensional vector $(1, \dots, 1)$. Moreover,

$$\mathcal{Q}_\alpha(x) = \mathcal{Q}(x) \qquad \text{if } \; Tx \in \alpha + \mathbb{Z}^{m_2}.$$

(b) *For all $\alpha \in [0,1)^{m_2}$ and $x \in \mathbb{R}^{n_1}$,*

$$\begin{aligned} \mathcal{Q}_\alpha(x) &= \sum_{i=1}^{m_2} \left(q_i^+ \mathbb{E}_{\psi_{\alpha_i}} \left[(\psi_{\alpha_i} - T_i x)^+ \right] + q^- \mathbb{E}_{\psi_{\alpha_i}} \left[(\psi_{\alpha_i} - T_i x)^- \right] \right) \\ &\qquad + \sum_{i=1}^{m_2} \frac{q_i^+ q_i^-}{q_i^+ + q_i^-}, \end{aligned}$$

where $\psi_{\alpha_i} = \lceil \xi_i \rceil_{\alpha_i} - \eta_i$, $i = 1, \dots, m_2$, and η_i is a random variable, independent of all other random variables, such that

$$\Pr\{\eta_i = 0\} = \frac{q_i^+}{q_i^+ + q_i^-}, \qquad \Pr\{\eta_i = 1\} = \frac{q_i^-}{q_i^+ + q_i^-}.$$

(c) *Independent of the distribution type of ξ it holds $\|\mathcal{Q}_\alpha - \mathcal{Q}\|_\infty \le 1$.*
Assume that for $i = 1, \dots, m_2$ the pdf f_i is contained in $\mathcal{F}$. Then for all $\alpha \in \mathbb{Z}^{m_2}$,

$$\|\mathcal{Q}_\alpha - \mathcal{Q}\|_\infty \le \sum_{i=1}^{m_2} (q_i^+ + q_i^-) \frac{|\Delta| f_i}{4}.$$

In particular, if each pdf f_i is unimodal then

$$\|\mathcal{Q}_\alpha - \mathcal{Q}\|_\infty \le \sum_{i=1}^{m_2} (q_i^+ + q_i^-) \frac{f_i(\nu_i)}{2},$$

where ν_i is the mode of the distribution of ξ_i.

(d) *Let $\alpha, \beta \in [0,1)^{m_2}$. For $i = 1, \dots, m_2$, define $\xi_{\alpha\beta_i}$ to be the continuous random variable with piecewise constant pdf $f_{\alpha\beta_i}$ given by*

$$f_{\alpha\beta_i}(s) = \frac{f_{\alpha_i}(s) + f_{\beta_i}(s)}{2}, \qquad s \in \mathbb{R},$$

where f_{α_i} (f_{β_i}) denotes the right-continuous pdf of the α_i (β_i)-approximation of ξ_i. Then

$$\mathcal{Q}_{\alpha\beta}(x) = \sum_{i=1}^{m_2} \left(q_i^+ \mathbb{E}_{\xi_{\alpha\beta_i}} \left[\lceil \xi_{\alpha\beta_i} - T_i x \rceil^+ \right] + q_i^- \mathbb{E}_{\xi_{\alpha\beta_i}} \left[\lfloor \xi_{\alpha\beta_i} - T_i x \rfloor^- \right] \right)$$

is a polyhedral convex function.

(e) *For all $\alpha, \beta \in [0,1)^{m_2}$ and $x \in \mathbb{R}^{n_1}$,*

$$\begin{aligned} \mathcal{Q}_{\alpha\beta}(x) &= \sum_{i=1}^{m_2} \left(q_i^+ \mathbb{E}_{\psi_{\alpha\beta_i}} \left[(\psi_{\alpha\beta_i} - T_i x)^+ \right] + q_i^- \mathbb{E}_{\psi_{\alpha\beta_i}} \left[(\psi_{\alpha\beta_i} - T_i x)^- \right] \right) \\ &\qquad + \sum_{i=1}^{m_2} \frac{q_i^+ q_i^-}{q_i^+ + q_i^-}, \end{aligned}$$

where, for $i = 1, \ldots, m_2$,

$$\psi_{\alpha\beta_i} = \begin{cases} \lceil \xi_i \rceil_{\alpha_i} \text{ w.p. } p_{i0}/2; \\ \lfloor \xi_i \rfloor_{\alpha_i} \text{ w.p. } p_{i1}/2; \\ \lceil \xi_i \rceil_{\beta_i} \text{ w.p. } p_{i0}/2; \\ \lfloor \xi_i \rfloor_{\beta_i} \text{ w.p. } p_{i1}/2, \end{cases}$$

with $p_{i0} := q_i^+/(q_i^+ + q_i^-)$ *and* $p_{i1} := q_i^-/(q_i^+ + q_i^-)$.

(f) *If, for* $i = 1, \ldots, m_2$, $f_i \in \mathcal{F}$ *and* $|\alpha_i - \beta_i| = 1/2$, *then*

$$\|\mathcal{Q}_{\alpha\beta} - \mathcal{Q}\|_\infty \quad \leq \quad \sum_{i=1}^{m_2} (q_i^+ + q_i^-) \frac{|\Delta| f_i}{8}.$$

Moreover, this uniform error bound can not be reduced by using other convex combinations of α*-approximations.*

We see that the best α-approximation (that is, with the lowest error bound) of the function $\mathcal{Q}$ is obtained by using compound pdfs $f_{\alpha\beta_i}$, $i = 1, \ldots, m_2$, as defined under (d) in the theorem above. However, unlike the case with simple pdf f_{α_i}, we do not know the points where the functions $\mathcal{Q}$ and $\mathcal{Q}_{\alpha\beta}$ coincide. Below we will see that precisely this information is used for comparing the optimal values of a simple integer recourse problem and its α-approximations.

In any case, the main conclusion is that for each $\alpha \in [0, 1)^{m_2}$ the associated α-approximation of the simple integer recourse model, belonging to the family of convex approximations given by

$$\inf_x \{cx + \mathcal{Q}_\alpha(x) : Ax = b,\ x \in \mathbb{R}_+^{n_1}\}, \quad \alpha \in [0, 1)^{m_2}, \tag{21}$$

is equivalent (up to a known constant in the objective function) to a continuous simple recourse problem with random right-hand side vector $\psi_\alpha = (\psi_{1,\alpha_1}, \ldots, \psi_{m_2,\alpha_{m_2}})$ with known discrete distribution. Consequently, it can be solved by existing special purpose algorithms for such problems. Obviously, the same is true for any approximation based on compound pdfs.

8.1 Optimal values of a SIR model and its α-approximations

We conclude with some results on the relation between the optimal values of the original simple integer recourse problem (1) and the family of convex approximations given in (21). For convenience, we will denote these problems by $\mathcal{P}$ and $\mathcal{P}_\alpha$, respectively. In fact, we will use this notation to denote the problem as well as its optimal value, but the actual meaning will be clear from the context.

Lemma 8.1 *Consider a simple integer recourse program* $\mathcal{P}$ *with continuously distributed right-hand side vector, and the family of convex* α*-approximations* $\mathcal{P}_\alpha$, $\alpha \in [0, 1)^{m_2}$. *Then*

(a) $\inf_\alpha \mathcal{P}_\alpha \leq \mathcal{P}$

(b) *If for some* $\hat{\alpha} \in [0, 1)^{m_2}$ *the problem* $\mathcal{P}_{\hat{\alpha}}$ *has an optimal solution* $x_{\hat{\alpha}}$ *such that* $Tx_{\hat{\alpha}} \in \hat{\alpha} + \mathbb{Z}^{m_2}$, *then* $\mathcal{P}_{\hat{\alpha}} \geq \mathcal{P}$.

(c) *If, for all* $\alpha \in [0,1)^{m_2}$*, the problem* $\mathcal{P}_\alpha$ *has an optimal solution* x_α *such that* $Tx_\alpha \in \alpha + \mathbb{Z}^{m_2}$*, then* $\inf_\alpha \mathcal{P}_\alpha = \mathcal{P}$.

PROOF. (a) The result is trivially true if problem $\mathcal{P}$ has no solution.
Let $\hat{x}$ be an optimal solution of $\mathcal{P}$. Define $\hat{\alpha} = (\hat{\alpha}_1, \ldots, \hat{\alpha}_{m_2})$, with $\hat{\alpha}_i = T_i\hat{x} - \lfloor T_i\hat{x} \rfloor$, $i = 1, 2, \ldots, m_2$. Then obviously $T\hat{x} \in \hat{\alpha} + \mathbb{Z}^{m_2}$, so that $\mathcal{Q}_{\hat{\alpha}}(\hat{x}) = \mathcal{Q}(\hat{x})$ by Theorem 8.1. In turn, this implies that $\mathcal{P}_{\hat{\alpha}} = \mathcal{P}$, which completes the proof.

(b) Immediate from Theorem 8.1 which implies that $\mathcal{P}_{\hat{\alpha}} = cx_{\hat{\alpha}} + \mathcal{Q}(x_{\hat{\alpha}})$, and the fact that $x_{\hat{\alpha}}$ is a feasible solution of problem $\mathcal{P}$.

(c) Immediate from (a) and (b).

□

It is clear that the practical implications of Lemma 8.1 are not very strong. However, it follows from Theorem 8.1 that for an arbitrary α the error of the approximation is small, as soon as the total variation of the marginal probability density functions is small enough. On the other hand, if these total variations are relatively high, then the choice of α can significantly influence the quality of the approximation. Fortunately, using their continuous simple recourse representation, the problems $\mathcal{P}_\alpha$ can be solved very efficiently, allowing to evaluate a large number of parameter values $\alpha \in [0,1)^{m_2}$ if necessary.

9. Concluding remarks

As stated in the introduction of this paper, simple integer recourse programs are convex only in exceptional cases. This claim is backed by giving a complete description of the small class of distributions that result in convex expected value functions. More importantly, the results also provide a foundation for obtaining convex approximations for the more common non-convex case.

Indeed, in case the function Q is non-convex for a given distribution with cdf F, we can construct a convex approximation of Q by replacing the original distribution by one with a pdf belonging to $\mathcal{C}$. As indicated by Lemma 3.3, the pdf generated by $G(s) = F(s - 1/2)$, $s \in \mathbb{R}$, is a possible candidate since it has the same mean value as F, and only a slightly increased variance (see Figure 4.1). However, although it is reasonable to expect that a good approximation of F yields a good approximation of Q, we have not been able to derive a non-trivial error bound for such approximations in general. Such an error bound can be derived for the class of α-approximations introduced in Section 5, in which case the corresponding pdf is generated by $G(s) = F(\lfloor s \rfloor_\alpha) := F(\lfloor s - \alpha \rfloor + \alpha)$ for any fixed $\alpha \in [0,1)$.

Moreover, we have shown that the resulting convex approximating problem is equivalent to a continuous simple recourse problem with explicitly known distribution of the right-hand side random parameters. Thus, instead of solving a non-convex simple integer recourse problem, we can resort to solving a continuous simple recourse problem, for which several efficient algorithms are available (see e.g. [8, 11]). This approach, in which the underlying distribution and the recourse structure are modified simultaneously to obtain an approximating problem which is easier to solve, can be applied also to other problem classes, see [16].

Finally, we mention that although the results presented in this paper have been available in research reports since 1997, they have not been published before. The main reason for this was that at the time it was believed – by referees and authors – that it would not be possible to generalize the results to more general recourse structures. However, in the mean time we have been able to obtain similar results for the general class of *complete* integer recourse models [17]. Recently, the same approach has yielded first results for mixed-integer recourse problems. Thus, looking back, it appears that the results reported in this paper have laid the foundations for an original approach to solving (mixed-)integer recourse problems.

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Appendix

9.1 Proof of Lemma 2.4

PROOF. The definitions of total increase and total decrease when applied to $\varphi := \varphi_1 - \varphi_2$ imply

$$\begin{aligned} &\Delta^+\varphi([s,\infty)) \le \varphi_2(s), \quad s \in \mathbb{R}_+, \\ &\Delta^-\varphi([s,\infty)) \le \varphi_1(s), \quad s \in \mathbb{R}_+, \end{aligned} \tag{22}$$

Indeed,

$$\begin{aligned} \Delta^+\varphi([s,\infty)) &\le \Delta^+\varphi_1([s,\infty)) + \Delta^+(-\varphi_2)([s,\infty)) \\ &= 0 + \Delta^-\varphi_2([s,\infty)) = \varphi_2(s), \end{aligned}$$

since φ_1 and φ_2 are nonnegative and nonincreasing; the proof of the second inequality in (22) is analogous. As a consequence, φ is of bounded variation on $[0,\infty)$:

$$|\Delta|\varphi([0,\infty)) = \Delta^+\varphi([0,\infty)) + \Delta^-\varphi([0,\infty)) \le \varphi_2(0) + \varphi_1(0) < \infty.$$

Since φ_1 is nonnegative and nonincreasing, it holds (see e.g. [**?**]) that

$$\begin{aligned} \int_0^\infty \varphi_i(s)\,ds &\le \sum_{k=0}^\infty \varphi_i(k) \le \int_0^\infty \varphi_i(s)\,ds + \varphi_i(0), \quad i = 1,2, \\ -\varphi_2(0) &\le \sum_{k=0}^\infty \varphi(k) - \int_0^\infty \varphi(s)\,ds \le \varphi_1(0). \end{aligned} \tag{23}$$

We will prove that (4) is true because of (23). Both give a lower and upper bound for the same difference of sum and integral. Because of (22), the bounds in (23) seem to be weaker. However, by replacing the generators φ_1 and φ_2 of φ by a 'minimal' one, we get (4). Indeed, define the functions $\tilde{\varphi}_i$, $i = 1, 2$, on $[0, \infty)$ by

$$\begin{aligned}
\tilde{\varphi}_1(s) &:= \Delta^-\varphi([s, \infty)), \qquad s \in \mathbb{R}_+, \\
\tilde{\varphi}_2(s) &:= \Delta^+\varphi([s, \infty)), \qquad s \in \mathbb{R}_+.
\end{aligned}$$

Obviously, $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are finite, nonnegative, nonincreasing functions on $[0, \infty)$ satisfying $0 \le \tilde{\varphi}_i \le \varphi_i$, $i = 1, 2$. Therefore they are integrable too. Moreover, $\tilde{\varphi} := \tilde{\varphi}_1 - \tilde{\varphi}_2 = \varphi$. This is a direct consequence of

$$-\tilde{\varphi}(s) = \Delta^+\varphi([s, \infty)) - \Delta^-\varphi([s, \infty)) = \lim_{t\to\infty} \varphi(t) - \varphi(s) = -\varphi(s), \quad s \in \mathbb{R},$$

where the second equality follows from the definitions of total increase and total decrease. As a consequence, $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ satisfy all the conditions we need for φ_1 and φ_2. Using them in (23) proves (4).

Finally, to prove (5) we use (4) to obtain

$$-\Delta^+\varphi([0, \infty)) - \varphi(0) \le \sum_{k=1}^{\infty} \varphi(k) - \int_0^{\infty} \varphi(s)\, ds \le \Delta^-\varphi([0, \infty)) - \varphi(0),$$

and the result now follows since

$$\begin{aligned}
-\Delta^+\varphi([0, \infty)) - \varphi(0) &= -\tilde{\varphi}_2(0) - \Big(\tilde{\varphi}_1(0) - \tilde{\varphi}_2(0)\Big) = -\tilde{\varphi}_1(0) = -\Delta^-\varphi([0, \infty)) \\
\Delta^-\varphi([0, \infty)) - \varphi(0) &= \tilde{\varphi}_1(0) - \Big(\tilde{\varphi}_1(0) - \tilde{\varphi}_2(0)\Big) = \tilde{\varphi}_2(0) = \Delta^+\varphi([0, \infty)).
\end{aligned}$$

□

9.2 Proof of Theorem 6.1

The proof of Theorem 6.1 relies on two alternative expressions for the difference $g_\alpha(z) - g(z)$, $z \in \mathbb{R}$, which we will derive first.

For all $z \in \mathbb{R}$ and $\alpha \in [0, 1)$, we have

$$\begin{aligned}
g_\alpha(z) - g(z) &= \sum_{k=0}^{\infty} \big(1 - F_\alpha(z + k)\big) - \sum_{k=0}^{\infty} \big(1 - F(z + k)\big) \\
&= \sum_{k=0}^{\infty} \big(F(z + k) - F_\alpha(z + k)\big) \\
&= \sum_{k=0}^{\infty} \int_{-\infty}^{z+k} \big(f(s) - f_\alpha(s)\big)\, ds.
\end{aligned}$$

In particular

$$g_\alpha(\lfloor z\rfloor_\alpha) - g(\lfloor z\rfloor_\alpha) = \sum_{k=0}^{\infty} \int_{-\infty}^{\lfloor z\rfloor_\alpha + k} \big(f(s) - f_\alpha(s)\big)\, ds.$$

Subtraction gives

$$g_\alpha(z) - g(z) = g_\alpha(\lfloor z \rfloor_\alpha) - g(\lfloor z \rfloor_\alpha) + \sum_{k=0}^{\infty} \int_{\lfloor z \rfloor_\alpha + k}^{z+k} \big(f(s) - f_\alpha(s)\big)\, ds.$$

From Lemma 6.1 we know that g_α and g coincide in $\lfloor z \rfloor_\alpha$, so that

$$g_\alpha(z) - g(z) = \sum_{k=0}^{\infty} \int_{\lfloor z \rfloor_\alpha + k}^{z+k} \big(f(s) - f_\alpha(s)\big)\, ds. \tag{24}$$

Moreover, it follows from the definition of f_α that for all $k \in \mathbb{Z}$

$$\int_{\lfloor z \rfloor_\alpha + k}^{z+k} \big(f(s) - f_\alpha(s)\big)\, ds + \int_{z+k}^{\lceil z \rceil_\alpha + k} \big(f(s) - f_\alpha(s)\big)\, ds = 0,$$

so that we also have the alternative expression

$$g_\alpha(z) - g(z) \quad = \quad -\sum_{k=0}^{\infty} \int_{z+k}^{\lceil z \rceil_\alpha + k} \big(f(s) - f_\alpha(s)\big)\, ds. \tag{25}$$

A further simplification of the expressions (24) and (25) is obtained from the fact that f_α is constant on every interval $[\alpha + k, \alpha + k + 1)$, $k \in \mathbb{Z}$. Hence,

$$\begin{aligned}
\sum_{k=0}^{\infty} \int_{\lfloor z \rfloor_\alpha + k}^{z+k} f_\alpha(s)\, ds &= \sum_{k=0}^{\infty} (z - \lfloor z \rfloor_\alpha) \int_{\lfloor z \rfloor_\alpha + k}^{\lfloor z \rfloor_\alpha + k + 1} f_\alpha(s)\, ds \\
&= (z - \lfloor z \rfloor_\alpha) \int_{\lfloor z \rfloor_\alpha}^{\infty} f_\alpha(s)\, ds \\
&= (z - \lfloor z \rfloor_\alpha) \big(1 - F_\alpha(\lfloor z \rfloor_\alpha)\big) \\
&= (z - \lfloor z \rfloor_\alpha) \big(1 - F(\lfloor z \rfloor_\alpha)\big).
\end{aligned}$$

Similarly,

$$\sum_{k=0}^{\infty} \int_{z+k}^{\lceil z \rceil_\alpha + k} f_\alpha(s)\, ds = (\lceil z \rceil_\alpha - z) \big(1 - F(\lfloor z \rfloor_\alpha)\big).$$

With respect to the terms in (24) and (25) in which $f(s)$ occurs, it is convenient to bring the summation inside the integral, giving

$$\sum_{k=0}^{\infty} \int_{\lfloor z \rfloor_\alpha + k}^{z+k} f(s)\, ds = \sum_{k=0}^{\infty} \int_{\lfloor z \rfloor_\alpha}^{z} f(s+k)\, ds = \int_{\lfloor z \rfloor_\alpha}^{z} \sum_{k=0}^{\infty} f(s+k)\, ds.$$

Similarly,

$$\sum_{k=0}^{\infty} \int_{z+k}^{\lceil z \rceil_\alpha + k} f(s)\, ds = \int_{z}^{\lceil z \rceil_\alpha} \sum_{k=0}^{\infty} f(s+k)\, ds.$$

Using these reformulations, (24) and (25) can be rewritten to (26) and (27), respectively:

$$g_\alpha(z) - g(z) = \int_{\lfloor z \rfloor_\alpha}^{z} \sum_{k=0}^{\infty} f(s+k)\, ds - (z - \lfloor z \rfloor_\alpha)\big(1 - F(\lfloor z \rfloor_\alpha)\big) \tag{26}$$

and

$$g_\alpha(z) - g(z) = (\lceil z \rceil_\alpha - z)\big(1 - F(\lfloor z \rfloor_\alpha)\big) - \int_{z}^{\lceil z \rceil_\alpha} \sum_{k=0}^{\infty} f(s+k)\, ds. \tag{27}$$

Below we use both (26) and (27) to prove Theorem 6.1, which states a bound on the error $|g_\alpha - g|$ that is uniform in α and z.

PROOF (THEOREM 6.1). The outline of the proof is as follows. First we derive two lower bounds for $g_\alpha - g$ using the expressions (26) and (27), to be followed by the corresponding upper bounds. Subsequently, the result follows by taking the maximum and minimum of these bounds, respectively.

First we determine a lower bound for $g_\alpha(z) - g(z)$ based on (26). By Lemma 2.5 we have

$$\sum_{k=0}^{\infty} f(s+k) \geq 1 - F(s-1) - \frac{|\Delta| f}{2}, \quad s \in \mathbb{R}.$$

It follows from (26) that

$$\begin{aligned} g_\alpha(z) - g(z) &= \int_{\lfloor z \rfloor_\alpha}^{z} \sum_{k=0}^{\infty} f(s+k)\, ds - (z - \lfloor z \rfloor_\alpha)\big(1 - F(\lfloor z \rfloor_\alpha)\big) \\ &\geq \int_{\lfloor z \rfloor_\alpha}^{z} \left(1 - F(s-1) - \frac{|\Delta| f}{2}\right) ds - (z - \lfloor z \rfloor_\alpha)\big(1 - F(\lfloor z \rfloor_\alpha)\big) \\ &\geq (z - \lfloor z \rfloor_\alpha)\left(1 - F(z-1) - \frac{|\Delta| f}{2}\right) - (z - \lfloor z \rfloor_\alpha)\big(1 - F(\lfloor z \rfloor_\alpha)\big) \\ &\geq -(z - \lfloor z \rfloor_\alpha)\frac{|\Delta| f}{2}, \end{aligned} \tag{28}$$

where we used that F is non-decreasing to obtain the second and third inequality; the last inequality then follows from $(z - \lfloor z \rfloor_\alpha)(-F(z-1) + F(\lfloor z \rfloor_\alpha)) \geq 0$ since $z - 1 \leq \lfloor z \rfloor_\alpha \leq z$.

To obtain a second lower bound we use Lemma 2.5, giving

$$\sum_{k=0}^{\infty} f(s+k) \leq 1 - F(s) + \frac{|\Delta| f}{2}, \quad s \in \mathbb{R}.$$

Hence (27) leads to

$$g_\alpha(z) - g(z) = (\lceil z \rceil_\alpha - z)\big(1 - F(\lfloor z \rfloor_\alpha)\big) - \int_{z}^{\lceil z \rceil_\alpha} \sum_{k=0}^{\infty} f(s+k)\, ds$$

$$\begin{aligned}
&\geq \quad (\lceil z \rceil_\alpha - z)\left(1 - F(\lfloor z \rfloor_\alpha)\right) - \int_z^{\lceil z \rceil_\alpha} \left(1 - F(s) + \frac{|\Delta| f}{2}\right) ds \\
&\geq \quad (\lceil z \rceil_\alpha - z)\left(1 - F(\lfloor z \rfloor_\alpha)\right) - (\lceil z \rceil_\alpha - z)\left(1 - F(z) + \frac{|\Delta| f}{2}\right) \\
&\geq \quad -(\lceil z \rceil_\alpha - z)\,\frac{|\Delta| f}{2}. \qquad (29)
\end{aligned}$$

Since $g_\alpha(z) - g(z)$ is greater or equal to the pointwise maximum of (28) and (29) we obtain

$$\begin{aligned}
g_\alpha(z) - g(z) &\geq \max\left\{ - (z - \lfloor z \rfloor_\alpha), -(\lceil z \rceil_\alpha - z)\right\} \frac{|\Delta| f}{2} \\
&= -\min\left\{z - \lfloor z \rfloor_\alpha, \lceil z \rceil_\alpha - z\right\} \frac{|\Delta| f}{2} \\
&\geq -\frac{|\Delta| f}{4}.
\end{aligned}$$

The upper bounds for $g_\alpha - g$ are found in a similar way. Equation (26) together with Lemma 2.5 results in

$$\begin{aligned}
g_\alpha(z) - g(z) &\leq (z - \lfloor z \rfloor_\alpha)\left(1 - F(\lfloor z \rfloor_\alpha) + \frac{|\Delta| f}{2}\right) \\
&\qquad\qquad - (z - \lfloor z \rfloor_\alpha)\left(1 - F(\lfloor z \rfloor_\alpha)\right) \\
&\leq (z - \lfloor z \rfloor_\alpha)\,\frac{|\Delta| f}{2},
\end{aligned}$$

and (27) together with Lemma 2.5 results in

$$\begin{aligned}
g_\alpha(z) - g(z) &\leq (\lceil z \rceil_\alpha - z)\left(1 - F(\lfloor z \rfloor_\alpha)\right) \\
&\qquad\qquad - (\lceil z \rceil_\alpha - z)\left(1 - F(\lceil z \rceil_\alpha - 1) - \frac{|\Delta| f}{2}\right) \\
&\leq (\lceil z \rceil_\alpha - z)\,\frac{|\Delta| f}{2}.
\end{aligned}$$

We conclude that for all $z \in \mathbb{R}$

$$\begin{aligned}
g_\alpha(z) - g(z) &\leq \min\left\{z - \lfloor z \rfloor_\alpha, \lceil z \rceil_\alpha - z\right\} \frac{|\Delta| f}{2} \\
&\leq \frac{|\Delta| f}{4}.
\end{aligned}$$

This completes the proof for the general case.

Finally, the result for unimodal distributions with mode ν follows immediately from the observation that $|\Delta| f = 2 f(\nu)$ in this case. $\square$